Properties of Definite Integrals: Definite Integral Definition

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We will be exploring some of the important **properties of definite integrals** and their proofs in this article to get a better understanding. Integration is the estimation of an integral. It is just the opposite process of differentiation. Integrals in maths are used to find many useful quantities such as areas, volumes, displacement, etc. There are two types of Integrals namely, definite integral and indefinite integral. Here, we will learn about definite integrals and its properties, which will help to solve integration problems based on them.

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Definite Integral Definition

If an integral has upper and lower limits, it is called a Definite Integral. There are many definite integral formulas and properties. Definite Integral is the difference between the values of the integral at the specified upper and lower limit of the independent variable. It is represented as;

$$\int_{a}^{b} f(x)\,dx$$

Definite Integral Properties

Following is the list of definite integrals in the tabular form which is easy to read and understand.

Properties	Description
Property 1	$\int_{p}^{q} f(a)\,da = \int_{p}^{q} f(t)\,dt$
Property 2	$\int_{p}^{q} f(a)\,da = -\int_{p}^{q} f(a)\,da,$ Also $\int_{p}^{q} f(a)\,da = 0$

Property 3	$\displaystyle\int_{p}^{q} f(a)\,da = \int_{p}^{r} f(a)\,da + \int_{r}^{q} f(a)\,da$
Property 4	$\displaystyle\int_{p}^{q} f(a)\,da == \int_{p}^{q} f(p+q-a)\,d(a)$
Property 5	$\displaystyle\int_{o}^{p} f(a)\,da = \int_{o}^{p} f(p-a)\,da =$
Property 6	$\displaystyle\int_{0}^{2p} f(a)\,da = \int_{0}^{p} f(a)\,da + \int_{0}^{p} f(2p-a)\,da \cdots if\, f(2p-a) = f(a)$
Property 7	2 parts $\displaystyle\int_{0}^{2} f(a)\,da = 2\int_{0}^{a} f(a)\,da \cdots if\, f(2p-a) = f(a)$ $\displaystyle\int_{0}^{2} p\, f(a)\,da = 0 \cdots if\, f(2p-a) = -f(a)$
Property 8	2 parts $\displaystyle\int_{-p}^{p} f(a)\,da = 2\int_{0}^{p} f(a)\,da \cdots if\, f(-a) = f(a)$ or it's an even function $\displaystyle\int_{-p}^{p} f(a)\,da = 0 \cdots if\, f(2p-a) = -f(a)$ or it's an odd function

Definite Integral Properties

Properties of Definite Integrals Proofs

Property 1:

$$\int_{p}^{q} f(a)\,da = \int_{p}^{q} f(t)\,dt$$

This is the simplest property as only a is to be substituted by t, and the desired result is obtained.

Property 2:

$$\int_{p}^{q} f(a)\,da = -\int_{p}^{q} f(a)\,da \text{ , also}$$

$$\int_{p}^{q} f(a)\,da = 0$$

Suppose $I = \int_{p}^{q} f(a)\, da$

If f' is the anti-derivative of f, then use the second fundamental theorem of calculus, to get

$I = f'(q) - f'(p) = -\left[f'(p) - f'(q) \right] = -\int_{p}^{q} f(a)\, da$ Also, if $p = q$ then

$I = f'(q) - f'(p) = f'(p) - f'(p) = 0$

Hence, $\int_{a}^{a} f(a)\, da = 0$.

Property 3: $\int_{p}^{q} f(a)\, da = \int_{p}^{r} f(a)\, da + \int_{r}^{q} f(a)\, da$ If f' is the anti-derivative of f, then use the second

fundamental theorem of calculus, to get;

$$\int_{p}^{q} f(a)\, da = f'(q) - f'(p) \ldots (1)$$

$$\int_{p}^{r} f(a)\, da = f'(r) - f'(p) \ldots (2)$$

$$\int_{r}^{q} f(a)\, da = f'(q) - f'(r) \ldots (3)$$

Let's add equations (2) and (3) , to get

$$\int_{p}^{q} f(a)\, da f(a)\, da + \int_{r}^{q} f(a)\, da f(a)\, da = f'(r) - f'(p) + f'(q)$$

$$= f'(q) - f'(p) = \int_{p}^{q} f(a)\, da$$

Property 4: $\int_{p}^{q} f(a)\, da = \int_{p}^{q} f(p + q - a)\, da$

Let, $t = (p + q - a), or\, a = (p + q - t)$, so that $dt = -\, da \ldots (4)$

Also, note that when $a = p, t = q$ and when $a = q, t = p$. So, $\int_{p}^{q}$ wil be replaced by $\int_{p}^{q}$ when we

replace a by t. Therefore,

$$\int_{p}^{q} f(a)\, da = -\int_{p}^{q} f(p + q - t)\, dt \text{ from equation (4)}$$

From property 2, we know that $\int_{p}^{q} f(a)\, da = -\int_{p}^{q} f(a)\, da$. Use this property, to get

$$\int_p^q f(a)\,da = \int_p^q f(p+q-t)\,da$$

Now use property 1 to get

$$\int_p^q f(a)\,da = \int_p^q f(p+q-a)\,da$$

Property 5: $\displaystyle\int_0^p f(a)\,da = \int_0^p f(p-a)\,da$

Let, $t = (p-a)\ or\ a = (p-t)$, so that $dt = -da \ldots (5)$

Also, observe that when $a = 0, t = p$ and when $a = p, t = 0.$ So, $\displaystyle\int_0^p$ will be replaced by $\displaystyle\int_0^p$ when we replace a by t. Therefore,

$$\int_0^p f(a)\,da = -\int_p^0 f(p-t)\,da \cdots \text{ from equation (5)}$$

From Property 2, we know that $\displaystyle\int_p^q f(a)\,da = -\int_p^q f(a)\,da$ Using this property, we get

$$\int_0^p f(a)\,da = \int_0^p f(p-t)\,dt$$

Next, using Property 1, we get

$$\int_0^a f(a)\,da = \int_0^p f(p-a)\,da$$

Property 6: $\displaystyle\int_0^{2p} f(a)\,da = \int_0^p f(a)\,da + \int_0^p f(2p-a)\,da$

From property 3, we know that

$$\int_p^q f(a)\,da = \int_p^r f(a)\,da + \int_r^q f(a)\,da$$

Therefore, $\displaystyle\int_0^{2p} f(a)\,da = \int_0^p f(a)\,da + \int_p^{2p} f(a)\,da = I_1 + I_2 \cdots (6)$

Where, $I_1 = \displaystyle\int_0^p f(a)\,da$ and $I_2 \displaystyle\int_p^{2p} f(a)\,da$

Let, $t = (2p-a)\ or\ a = (2p-t)$, so that $dt = -da \ldots (7)$

Also, note that when $a = p, t = p$, and when $a = 2p, t = 0$. Hence, $\int_a^0$ when we replace a by t.

Therefore,

$$I_2 = \int_0^{2p} f(a)\, da = -\int_p^0 f(2p - 0)\, da \cdots \text{ from equation (7)}$$

From Property 2, we know that $\int_p^q f(a)\, da = -\int_p^q f(a)\, da$. Using this property, we get

$$I2 = \int_0^p f(2p - t)\, dt$$

Next, using Property 1, we get

$$I_2 \int_0^a f(a)\, da + \int_0^a f(2p - a)\, da$$

Replacing the value of I_2 in equation (6) , we get

$$\int_0^{2p} f(a)\, da = \int_0^p f(a)\, da + \int_0^p f(2p - a)\, da$$

Property 7:

$$\int_0^{2a} f(a)\, da = 2\int_2^a f(a)\, da \cdots iff(2p - a) = f(a) \text{ and } \int_0^{2a} f(a)\, da = 0 \cdots iff(2p - a) = -f(a)$$

we know that

$$\int_0^{2p} f(a)\, da = \int_0^p f(a)\, da + \int_0^p f(2p - a)\, da \cdots (8)$$

Now, if $f(2p - a) = f(a),$ then equation (8) becomes

$$\int_0^{2p} f(a)\, da = \int_0^p f(a)\, da + \int_0^p f(a)\, da$$

$$= 2\int_0^p f(a)\, da$$

And, if $f(2p - a) = -f(a),$ then equation (8) becomes

$$\int_0^{2p} f(a)\, da = \int_0^p f(a)\, da - \int_0^p f(a)\, da$$

Property 8: $\displaystyle\int_{p}^{q} f(a)\,da = \int_{0}^{p} f(a)\,da \cdots if f(-a) = f(a)$ **or it is an even function and**

$\displaystyle\int_{-a}^{a} f(a)\,da = 0 \cdots if f(-a) = f(a)$ **or it is an odd function**.

Using Property 3, we have

$$\int_{-p}^{p} f(a)\,da = \int_{-a}^{0} f(a)\,da + \int_{0}^{p} f(a)\,da = I_1 + I_2 \cdots (9)$$

Where, $\displaystyle I_1 = \int_{-a}^{0} f(a)\,da, I_2 \int_{0}^{p} f(a)\,da$

Consider I_1

Let, $t = -a \, or \, a = -t, so\,that\,dt = -dx \ldots (10)$

Also, observe that when $a = -p, t = p, when\, a = 0, t = 0$. Hence, $\displaystyle\int_{-a}^{0}$ will be replaced by $\displaystyle\int_{a}^{0}$ when we

replace a by t. Therefore,

$$I_1 = \int_{-a}^{0} f(a)\,da = \int_{a}^{0} f(-a)\,da \cdots from\,equation\,(10)$$

From Property 2, we know that $\displaystyle\int_{p}^{q} f(a)\,da = -\int_{p}^{q} f(a)\,da$, use this property to get,

$$I_1 = \int_{-p}^{0} f(a)\,da = \int_{0}^{p} f(-a)\,da$$

Next, using Property 1, we get

$$I_1 = \int_{-p}^{q} f(a)\,da = \int_{0}^{p} f(-a)\,da$$

Replacing the value of I_2 in equation (9) , we get

$$\int_{p}^{q} f(a)\,da = I_1 + I_2 = \int_{0}^{p} f(a)\,da + \int_{0}^{p} f(a)\,da = 2\int_{0}^{p} f(a)\,da \cdots (11)$$

Now, if 'f' is an even function, then $f(-a) = f(a)$. Therefore, equation (11) becomes

$$\int_{-p}^{p} f(a)\,da = \int_{0}^{p} f(a)\,da + \int_{0}^{p} f(a)\,da = 2\int_{0}^{p} f(a)\,da$$

And, if 'f' is an odd function, then $f(-a) = -f(a)$. Therefore, equation (11) becomes

$$\int_{-p}^{p} f(a)\, da = -\int_{0}^{a} f(a)\, da + \int_{0}^{p} f(a)\, da$$

Now, let us evaluate Definite Integral through a problem sum.

Example

Question: Evaluate $\displaystyle\int_{-1}^{2} f\left(a^3 - a\right) da$

Solution: Observe that, $\left(a^3 - a\right) \geqslant 0$ on $[-1, 0]$, $\left(a^3 - a\right) \leqslant 0$ on $[0, 1]$ and $\left(a^3 - a\right) \geqslant 0$ on $[1, 2]$

Hence, using Property 3, we can write

$$\int_{-1}^{2} f\left(a^3 - a\right) da = \int_{-1}^{0} f\left(a^3 - a\right) da + \int_{0}^{1} f\left(a^3 - a\right) da + \int_{1}^{2} f\left(a^3 - a\right) da = \int_{-1}^{0} f\left(a^3 - a\right) da + \int_{0}^{1} f\left(a - a^3\right) da + \int_{1}^{2} f\left(a^3 - a\right) da$$

$$\int_{-1}^{0} f\left(a^3 - a\right) da + \int_{0}^{1} f\left(a - a^3\right) da + \int_{1}^{2} f\left(a^3 - a\right) da$$

Solving the integrals, we get

$$\int_{-1}^{2} f\left(a^3 - a\right) da = \left[\frac{x^4}{4} - \left(\frac{x^2}{2}\right)\right] - 10 + \left[\left(\frac{x^2}{2} - \left(\frac{x^4}{4}\right)\right) 01\right] + \left[\frac{x^4}{4} - \left(\frac{x^2}{2}\right)\right] 12$$

$$= -\left[\frac{1}{4} - \frac{1}{2}\right] + \left[-\frac{1}{4}\right] + [4 - 2] - \left[\frac{1}{4} - \frac{1}{2}\right] = \frac{11}{4}$$