FlexiPrep

CBSE Class 11-Mathematics: Mathematical Induction Assignments (For CBSE, ICSE, IAS, NET, NRA 2022)

Prove the Following using Principle of Mathematical induction

1) Prove that for any positive integer number n, $n^3 + 2n$ is divisible by 3

2) Prove that

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for all positive integers n.

3) For every $n \in N, 2n^3 + 3n^2 + n$ is divisible by 6 .

4) Prove by induction that

$$1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet \bullet n \bullet (n+1) = n(n+1)(n+2)/3$$

5) For every $n \geqslant 2, n^3 - n$ is multiple of 6

6) For every $n \geqslant 7, 3^n \geqslant n!$

7) For all $n \geqslant 1$

$$(1+x)^n + 1 \text{ nx}$$

Where $(1+x) > 0$

8) If $n \in N$, then $1 \bullet 3 + 2 \bullet 4 + 3 \bullet 5 + 4 \bullet 6 + \bullet \bullet \bullet + n(n+2) = n(n+1)(2n+7)/6$

9) Prove that $3 + 3^2 + 3^3 + 3^4 + \bullet \bullet \bullet + 3^n = \left(3^{n+1} - 3\right)/2$ for every $n \in N$.

10) Prove that $1/1 + 1/4 + 1/9 + \bullet \bullet \bullet + 1 + 1/n^2 \leqslant 2 - 1/n$

11) For all $n > 1, 8^n - 3^n$ is divisible by 5 .

Solution to Problem 1:

Let Statement $P(n)$ is defined by

$n^3 + 2n$ is divisible by 3

Step 1: Basic Step

We first show that p (1) is true. Let $n = 1$ and calculate $n^3 + 2n$

$$1^3 + 2(1) = 3$$

3 is divisible by 3

Hence p (1) is true.

STEP 2: Inductive Hypothesis

We now assume that $p(k)$ is true

$k^3 + 2k$ is divisible by 3 is equivalent to

$k^3 + 2k = 3B$, where B is a positive integer.

Step 3: Inductive Steps

We now consider the algebraic expression $(k+1)^3 + 2(k+1)$; expand it and group like terms

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 5k + 3$$

$$= \left[k^3 + 2k\right] + \left[3k^2 + 3k + 3\right]$$

$$= 3B + 3\left[k^2 + k + 1\right] = 3\left[B + k^2 + k + 1\right]$$

Hence $(k+1)^3 + 2(k+1)$ is also divisible by 3 and therefore statement $P(k+1)$ is true.

Solution to Problem 2:

Statement $P(n)$ is defined by

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n+1)^2/4$$

Step 1: Basic Step

We first show that $p(1)$ is true.

Left Side $= 1^3 = 1$

Right Side $= 1^2(1+1)^2/4 = 1$

Hence $p(1)$ is true.

STEP 2: Inductive Hypothesis

We now assume that $p(k)$ is true

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = k^2(k+1)^2/4$$

Step 3: Inductive Steps

Add $(k+1)^3$ to both sides

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = k^2(k+1)^2/4 + (k+1)^3$$

Factor $(k+1)^2$ on the right side

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$$

Set to common denominator and group

$$= (k+1)^2 \left[k^2 + 4k + 4\right]/4$$

$$= (k+1)^2 \left[(k+2)\,2\right]/4$$

We have started from the statement $P(k)$ and have shown that

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (k+1)^2 \left[(k+2)^2\right]/4$$

Which is the statement $P(k+1)$.

Solution to Problem 3:

Let $P(n)$

$2n^3 + 3n^2 + n$ is divisible by 6

Step 1: Basic Step

$P(1)$ is just that $2 + 3 + 1$ is divisible by 6, which is trivial.

STEP 2: Inductive Hypothesis

We now assume that $P(k)$ is true

i.e.. ,

$2k^3 + 3k^2 + k$ is divisible by 6

Step 3: Inductive Steps

We must prove $P(k+1)$

Now

$$2(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= 2\left(k^3 + 3k^2 + 3k + 1\right) + 3\left(k^2 + 2k + 1\right) + (k+1)$$

$$= \left(2k^3 + 3k^2 + k\right) + \left(6k^2 + 6k + 2 + 6k + 3 + 1\right)$$

$$= \left(2k^3 + 3k^2 + k\right) + 6\left(k^2 + 2k + 1\right)$$

The first term is divisible by 6 since $P(k)$ is true and the second term is a multiple of 6. Hence, the last quantity is divisible by 6

Solution to Problem 4:

Statement $P(n)$ is defined by

$$1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet \bullet n \bullet (n+1) = n(n+1)(n+2)/3$$

Step 1: Basic Step

We first show that $p(1)$ is true.

Left Side $= 1.2 = 2$

Right Side $= 1\,(1+1)\,(1+2)\,/3 = 2$

Hence $p\,(1)$ is true.

STEP 2: Inductive Hypothesis

We now assume that $p\,(k)$ is true

$$1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet \bullet k \bullet (k+1) = k\,(k+1)\,(k+2)\,/3$$

Step 3: Inductive Steps

We must prove $P\,(k+1)$

Now

$$1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet \bullet (k+1) \bullet [(k+1)+1) = (k+1)\,[(k+1)+1]\,[(k+1)+2]\,/3$$

Taking LHS

$$1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet \bullet (k+1) \bullet [(k+1)+1)$$

$$= 1 \bullet 2 + 2 \bullet 3 + 3 \bullet 4 + \bullet \bullet\, k\,(k+1) + (k+1) \bullet [(k+1)+1)$$

$$= k\,(k+1)\,(k+2)\,/3 + (k+) \bullet [(k+1)+1)$$

$$= k\,(k+1)\,(k+2)\,/3 + + (k+) \bullet (k+2)$$

$$= (k+1)\,(k+2)\left[\frac{k}{3}+1\right]$$

$$= (k+1)\,(k+2)\,(k+3)\,/3$$

Which is the statement $P\,(k+1)$.

Solution to Problem 6:

Let Statement $P\,(n)$ is defined by

For all $n > 7, n! > 3^n$

Step 1: Basic Step

Let $n = 7$

$$n! > 3^n$$

$$7! = 5040$$

$$3^7 = 2187$$

So $p\,(7)$ is true

STEP 2: Inductive Hypothesis

We now assume that $p\,(k)$ is true

That is, $k! > 3^k$

Step 3: Inductive Steps

Let $n = k + 1$.

Then:

$$(k+1)! = (k+1)\,k!$$

$$> (k+1)\,3^k$$

Now $k > 7$

So $(k+1) > 3$

$$> 3.\,3^k$$

> 3k + 1

Then $P(n)$ holds for $n = k + 1$, and thus for all $n > 7$

Solution to Problem 7:

Let Statement $P(n)$ is defined by

$$(1+x)^n \geqslant 1 + \text{nx}$$

Where $(1+x) > 0$

Step 1: Basic Step

Let $n = 1$

$$(1+x)^n \geqslant 1 + \text{nx}$$

$$(1+x) \geqslant 1 + x$$

Which is true

So $p(1)$ is true

STEP 2: Inductive Hypothesis

We now assume that $p(k)$ is true

$$(1+x)^k \geqslant 1 + \text{kx}$$

Step 3: Inductive Steps

Let $n = k + 1$.

Then:

$$(1+x)^{k+1} \geqslant 1 + (k+1)\,x$$

Taking the LHS

$$(1+x)^{k+1} = (1+x)(1+x)^k$$

Now from hypothesis we know that

$$(1+x)^k \geqslant 1 + \mathrm{kx}$$

Also $(1+x) > 0$

So $(1+x)^{k+1} \geqslant (1+x)(1+\mathrm{kx})$

$$\geqslant \left[1 + \mathrm{kx}^2 + (k+1)x\right]$$

Now kx2 is a positive quantity so we can say that

$$\geqslant [1 + (k+1)x]$$

Which is $P(k+1)$

Solution to Problem 11:

Let Statement $P(n)$ is defined by

For all $n > 1, 8^n - 3^n$ is divisible by 5 .

Step 1: Basic Step

Let $n = 1$.

Then the expression $8^n - 3^n$ evaluates to $81 - 31 = 8 - 3 = 5$, which is clearly divisible by 5.

STEP 2: Inductive Hypothesis

We now assume that $p(k)$ is true

That is, that $8^k - 3^k$ is divisible by 5 .

Step 3: Inductive Steps

Let $n = k + 1$.

Then:

$$8^{k+1} - 3^{k+1} = 8^{k+1} - 3 \times 8^k + 3 \times 8^k - 3^{k+1}$$

$$= 8^k(8-3) + 3\left(8^k - 3^k\right) = 8^k(5) + 3\left(8^k - 3^k\right)$$

The first term in $8^k(5) + 3\left(8^k - 3^k\right)$ has 5 as a factor (explicitly) , and the second term is divisible by 5 (by assumption) . Since we can factor a 5 out of both terms, then the entire expression, $8^k(5) + 3\left(8^k - 3^k\right) = 8^{k+1} - 3^{k+1}$, must be divisible by 5 .

Then $P(n)$ holds for $n = k + 1$, and thus for all $n > 1$.

Developed by: Mindsprite Solutions