MATHEMATICS

PAPER - I

SECTION A

1. Answer any four of the following:

(a) Let V be the vector space of polynomials in x with real coefficients of degree at most 2. Let t_1, t_2, t_3 be any 3 distinct real numbers. Define

$L_i : V \to \mathbb{R}$ by $L_i(f) = f(t_i)$, $i = 1, 2, 3$. Show that

(i) L_1, L_2, L_3 are linear functionals on V

(ii) $\{L_1, L_2, L_3\}$ is a basis for the dual space V* of V.

(10)

(b) In the notation of (*a*) above, find a basis $B = \{p_1, p_2, p_3\}$ for V which is dual to $\{L_1, L_2, L_3\}$ and also express each $P \in V$ in terms of elements of B.

(c) Let f be a function defined on [0, 1] by

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is irrational} \\ \frac{1}{q}, \text{ if } x = \frac{p}{q}, q \neq 0 \end{cases}$$

and p, q, are relatively prime positive integers. Show that f is continuous at each irrational point and discontinuous at each rational point $\frac{p}{q}$.

(10)

(d) Show that the function [x], where [x] denotes the greatest integer not greater than x, is integrable in [0, 3]. Also evaluate $\int_0^3 [x]dx$.

(10)

(e) Prove that the polar of one limiting point of a coaxial system of circles with respect to any circle of the system passes through the other limiting point.

(10)

2. (a) Let V be the vector space of polynomials in x with complex coefficients. Define

$T: V \to V$ by $(Tf)(x) = xf(x)$ and

$$U : V \to V \text{ by } U\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n-1} c_i + 1x^i$$

Find (i) ker T (ii) show U is linear (iii) show that UT = I and TU ≠ I, I = identity on V.

(10)

(b) *Let* $A=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$ and hence find the matrix A^8.

(10)

(c) Find the characteristic and minimal polynomials of

$$A=\begin{pmatrix}2 & 0 & 0\\ 1 & 2 & 0\\ 0 & 0 & -1\end{pmatrix}$$

and determine whether A is diagonalizable.

(10)

(d) Let $A=\begin{pmatrix}1 & 1 & 2\\ -1 & 2 & 1\\ 0 & 1 & 3\end{pmatrix}$

Find an invertible 3×3 matrix P such that $P^t AP = D$, D is diagonal matrix. Find D also.

(10)

3. (a) Examine the convergence of the integral

$$\int_0^1 x^{n-1} \log x dx$$

(10)

(b) Examine the function

$$f(x,y)\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

For continuity, partial derivability of the first order and differentiability at (0, 0).

(10)

(c) Find the maximum and minimum values of the function $f(x, y, z) = xy + 2z$ on the circle which is the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$.

(10)

(d) Find the volume of there region R lying below the plane $z = 3 + 2y$ and above the paraboloid $z = x^2 + y^2$.

(10)

4. (a) CP and CD are conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the orthocenter of the triangle CPD is the curve $2\left(b^2y^2+a^2x^2\right)^3=\left(a^2-b^2\right)^2\left(b^2y^2-a^2x^2\right)^2$

(13)

(b) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$ Find the equations of the other two.

(13)

(c) If the section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

Whose vertex is P, by the plan = 0 is a rectangular hyperbola, prove that the locus of P is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$.

(14)

SECTION B

5. Answer any four of the following:

(a) A tank of 100 liters capacity is initially full of water. Pure water is allowed to run into the tank at the rate of 1 liter per minute and at the same time salt water containing $\frac{1}{4}$ kg of salt per liter flows into the tank at the rate of 1 liter per minute. The mixture (there is perfect mixing in the tank at all times) flows out at the rate of 2 liters per minute. Form the differential equation and find the amount of salt in the tank after t minutes. Find this when t = 50 minutes.

(10)

(b) A constant coefficient differential equation has auxiliary equation expressible in factored form as

$P(m) = m^3 (m - 1)^2 (m^2 + 2m + 5)^2$.

What is the order of the differential equation and find its general solution.

(10)

(c) A particle rests in equilibrium under the reaction of two centers of forces which attract directly as the distance, their intensity being μ, μ', the particle is displaced slightly towards one of them, show that the time of a small oscillation is

$$T=\frac{2\pi}{\sqrt{\mu+\mu'}}$$

(10)

(d) A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

(10)

(e) Find an equation for the plane passing through the points $P_1(3, 1, -2)$, $P_2(-1, 2, 4)$, $P_3(2, -1, 1)$ by using vector method. (10)

6. (a) Solve $x^2\left(\frac{dy}{dx}\right)^2+y(2x+y)\frac{dy}{dx}+y^2=0$ (10)

(b) Using differential equations show that the system of confocal conics given by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \lambda \text{ real is}$$

Self-orthogonal.

(10)

(c) Solve

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

given that $y = e^{\sin^{-1}x}$ is one solution of this equation. (10)

(d) Find a general solution of $y'' + y = \tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ by variation of parameters. (10)

7. (a) A particle moves with central acceleration $(\mu u^2 + \lambda u^3)$ and the velocity of projection at distance R is V, show that the particle will ultimately go off to infinity if

$$V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}.$$

(13)

(b) A smooth parabolic wire is fixed with its axis vertical and vertex downwards and in it is placed a uniform rod of length $2l$ with its ends resting on wire. Show that, for equilibrium, the rod is either horizontal or makes with horizontal an angle θ given by $\cos^2\theta = \frac{2a}{l}$, $4a$ being the latus rectum of the parabola. (14)

(c) Prove that if volume v and V of two different substances balance in vacuum and volumes v′, V′ balance when weighed in a liquid, the densities of the substances and the liquid are as

$$\frac{v'-V'}{v} : \frac{v'-v'}{V} : \left(\frac{v'}{V} - \frac{V'}{V}\right)$$

(13)

8. (a) Prove that

$$\nabla \times \left(\nabla \times \overline{A}\right) = -\nabla^2\overline{A} + \nabla\left(\nabla.\overline{A}\right)$$

(10)

(b) If

$$\nabla.\overline{E}, \nabla.\overline{H}, \nabla \times \overline{E} = -\frac{\partial \overline{H}}{\partial t}, \nabla \times \overline{H} = \frac{\partial \overline{E}}{\partial t}$$

Show that $\overline{E}$ and $\overline{H}$ satisfy

$$\nabla^2\overline{u} = \frac{\partial^2\overline{u}}{\partial t^2}$$

(10)

(c) Given the space curve $x = t, y = t^2, z = \frac{2}{3}t^3$. Find (i) the curvature ρ (ii) the torsion τ. (10)

(d) If $F = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$, evaluate $\iint_s$ curl $\overline{F}.nds$, taken over the portion of the surface.

$x^2 + y^4 - z^2 - 2ax + uz = 0$ above the plane $z = 0$ and verify Stoke's theorem. (10)

MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts (4 x 10 = 40)

(a) Write the elements of the symmetric group S_3 of degree 3, prepare its multiplication table and find all normal subgroups of S_3

(b) Change the order of integration in the integral

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

and evaluate it.

(c) Compute the Taylor series around z = 0 and give the radius of convergence for $\frac{z}{z-1}$

(d) By graphical method solve the linear programming problem

$$\begin{aligned} \text{Maximize } Z &= 100X_1 + 40X_2 \\ \text{subject to } 5X_1 + 2X_2 &\le 1000 \\ 3X_1 + 2X_2 &\le 900 \\ X_1 + 2X_2 &\le 500 \\ X_1, X_2 &\ge 0. \end{aligned}$$

(e) If

$$a_n = \log\left(1+\frac{1}{n^2}\right) + \log\left(1+\frac{2}{n^2}\right) + \dots + \log\left(1+\frac{n}{n^2}\right)$$

Find $\lim_{n\to\infty} a_n$

2. (a) If every element of a group G is its own inverse, prove that the group G is abelian. Is the converse true? Justify your claim.

(14)

(b) Discuss the maxima and minima of $x^3 y^2(1-x-y)$.

(13)

(c) State the Weierstrass M-test for uniform convergence of an infinite series of functions. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n^{a}\left(1+nx^2\right)} \text{ with } a < 0$$

is uniformly convergent on $(-\infty, \infty)$. (13)

3. (a) Define a field and prove that every finite integral domain is a field.

(3 + 10)

(b) Show that the function $f(z)=\sqrt{|xy|}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied. (13)

(c) By using the Residue Theorem evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2} \text{ where } 0<a<1.$$

(14)

4. (a) Define a unique factorization domain. Show that $Z\left[\sqrt{-5}\right]$ is an integral domain which is not a unique factorization domain.

(2 + 10)

(b) Using simplex method solve the linear programming problem:

Maximize $Z = X_1 + X_2 + 3X_3$

subject to $3X_1 + 2X_2 + X_3 \le 3$

$2X_1 + X_2 + 2X_3 \le 2$

$X_1, X_2, X \ge 0.$

(13)

(c) A company has three plants A, B and C, three ware houses X, Y and Z. The number of units available at the plants is 60, 70, 80 and the demands at X, Y, Z are 50, 80, 80 respectively. The unit cost of the transportation is given in the following table:-

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum.

(14)

SECTION B

5. Answer any four parts:-

(4 × 10 = 40)

(a) Find the complete integral of the partial differential equation

$x^2p^2 + y^2q^2 = z^2.$

(b) Find Lagrange's interpolation polynomial $P_2(x)$ which satisfies.

$f(0) = P_2(0) = 1$

$f(-1) = P_2(-1) = 2$

$f(1) = P_2(1) = 3.$

Find $f(0.5)$.

(c) Convert the decimal number $(1479.25)_{10}$ to the binary and the hexadecimal numbers.

(d) Find the moment of inertia of the area bounded by $r^2 = a^2\cos 2\theta$ about its axis.

(e) Show that $\frac{x^2}{a^2}f(t)+\frac{y^2}{b^2}\cdot\frac{1}{f(t)}=1$ is a possible form of the bounding surface of a liquid.

6. (a) Solve by Charpit's method

$(p^2 + q^2)\,y = qz.$ (13)

(b) If $\varphi(x)$ is a continuous and bounded function of $-\infty < x < \infty$, prove that the function $u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-(x-\xi)^2/4kt} d\xi$ is a solution of the initial value problem:

$$u_t - k u_{xx} = 0, \ -\infty < x < \infty, \ t > 0$$
$$u(x,0) = \varphi(x) \text{ for } -\infty < x < \infty.$$

(c) Two equal masses m_1 and m_2 with $m_1 > m_2$ are suspended by a light string over a pulley of mass M and radius a. There is no slipping and the friction of axle may be neglected. If f be the acceleration, show that this is constant and if k^2 be the radius of gyration of the pulley about the axis, show that

$$k^2 = \frac{a^2}{Mf}\left[(g-f)m_1 - (g+f)m_2\right]$$

(13)

7. (a) Four equal rods, each of length 2a, are hinged at their ends so as to form a rhombus ABCD. The angles B and D are connected by an elastic string and the lowest end A rests on a horizontal plane whilst the end C slides on a smooth vertical wire passing through A. In the position of equilibrium the string is stretched to twice its natural length and the angle BAD is 2α. Show that the time of small oscillation about this position is

$$2\pi\left\{\frac{2a(1+3\sin^2\alpha)\cos\alpha}{(3g\cos 2\alpha)}\right\}^{\frac{1}{2}}$$

(14)

(b) If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = q\nabla^2 q$$

(c) By applying the Newton Raphson method to $f(x) = x^2 - a$ where $a > 0$, prove that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

For $n = 0, 1, 2, 3$ Apply this formula to find $\sqrt{2}$

8. (a) Write an algorithm for generating even integers ≤ 100. Also draw the flow chart which executes this algorithm.

(13)

(b) Applying Simpson's one-third rule compute the value of the definite integral

$$\int_{4}^{5.2} \log x \, dx$$

with h = 0.2 and estimate the error. (13)

(c) State the conditions under which the equations of motion can be integrated. Obtain Bernoulli's equation for the steady irrotational motion of an incompressible liquid. (14)