Solve all problems vectorially:

(1) Obtain the unit vectors perpendicular to each of $\bar{x}$ = (1, 2, - 1) and $\bar{y}$ = (1, 0, 2).

$$\left[\textbf{Ans :} \ \pm \left(\frac{4}{\sqrt{29}}, \ \frac{-3}{\sqrt{29}}, \ \frac{-2}{\sqrt{29}} \right) \right]$$

(2) If α is the angle between two unit vectors $\bar{a}$ and $\bar{b}$, then prove that
$| \bar{a} - \bar{b} \cos \alpha | = \sin \alpha$.

(3) If a vector $\bar{r}$ makes with X-axis and Y-axis angles of measures 45° and 60°
respectively, then find the measure of the angle which $\bar{r}$ makes with Z-axis.

[Ans: 60° or 120°]

(4) If $\bar{x}$ and $\bar{y}$ are non-collinear vectors of R^3, then prove that $\bar{x}$, $\bar{y}$ and $\bar{x} \times \bar{y}$
are non-coplanar.

(5) If the measure of angle between $\bar{x} = \bar{i} + \bar{j}$ and $\bar{y} = t\bar{i} - \bar{j}$ is $\dfrac{3\pi}{4}$, then find t.

[Ans: 0]

(6) Show that for any a $\in$ R, the directions (2, 3, 5) and (a, a + 1, a + 2) cannot be
the same or opposite.

(7) If θ is a measure of angle between unit vectors $\bar{a}$ and $\bar{b}$, prove that
$\sin \dfrac{\theta}{2} = \dfrac{1}{2} | \bar{a} - \bar{b} |$.

(8) If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are non-coplanar, prove that $\bar{x} + \bar{y}$, $\bar{y} + \bar{z}$ and $\bar{z} + \bar{x}$ are also
non-coplanar.

(9) Show that the vectors (1, 2, 1), (1, 1, 4) and (1, 3, - 2) are coplanar. Also express each of these vectors as a linear combination of the other two.

$$\text{Ans : } (1, 2, 1) = \frac{1}{2}(1, 1, 4) + \frac{1}{2}(1, 3, -2); \quad (1, 1, 4) = 2(1, 2, 1) - (1, 3, -2)$$
$$(1, 3, -2) = 2(1, 2, 1) - (1, 1, 4)$$

[Note: These vectors are collinear besides being coplanar. Hence, any vector of R^3 which is not collinear with them cannot be expressed as a linear combination of these vectors even if it is coplanar with them.]

(10) Show that (1, 1, 0), (1, 0, 1) and (0, 1, 1) are non-coplanar vectors. Also express any vector (x, y, z) of R^3 as a linear combination of these vectors.

$$\text{Ans : } (x, y, z) = \frac{x + y - z}{2}(1, 1, 0) + \frac{x - y + z}{2}(1, 0, 1) + \frac{y + z - x}{2}(0, 1, 1)$$

(11) Prove that an angle in a semi-circle is a right angle.

(12) Prove that the three altitudes in a triangle are concurrent.

(13) If A - P - B and $\dfrac{AP}{PB} = \dfrac{m}{n}$, then prove that for any point O in space,
$n(\overrightarrow{OA}) + m(\overrightarrow{OB}) = (m + n)\overrightarrow{OP}$.

(14) Prove that A (1, 5, 6), B (3, 1, 2) and C (4, - 1, 0) are collinear. Find also the ratio in which A divides $\overline{BC}$ from B.

[Ans: - 2 : 3]

(15) Find in which ratio and at which point does the XY-plane divide $\overline{AB}$ where A is (2, - 2, 1) and B is (1, 4, - 5).

$$\text{Ans : } 1 : 5 \text{ from A at } \left(\frac{11}{6}, -1, 0 \right)$$

(16) If A (0, - 1, - 1), B (16, - 3, - 3) and C (- 8, - 1, - 2) are given points, then find the point D (x, y, z) in space so that $\overrightarrow{AB} = \overrightarrow{CD}$.

[Ans: (8, - 3, - 4)]

(17) A (0, - 1, - 4), B (1, 2, 3) and C (5, 4, - 1) are given points. If is the foot of perpendicular from A on $\overline{BC}$, find its position vector.

[Ans: (3, 3, 1)]

(18) If the position vectors A, B, C of triangle ABC are $\bar{a}, \bar{b}, \bar{c}$ respectively, then show that the area of triangle ABC $= \frac{1}{2} \left| (\bar{a} \times \bar{b}) + (\bar{b} \times \bar{c}) + (\bar{c} \times \bar{a}) \right|$.

(19) Find the volume of a prism having a vertex at origin O and having coterminous edges $\overline{OA}, \overline{OB}, \overline{OC}$, where A is (4, 3, 1), B is (3, 1, 2) and C is (5, 2, 1).

[Ans: 10 cubic units]

(20) Find the volume of tetrahedron having vertices V (1, 1, 3), A (4, 3, 2), B (5, 2, 7) and C (6, 4, 8).

$$\left[\text{Ans :} \ \frac{14}{} \ \text{cubic units} \right]$$

(21) If the forces of magnitudes $\sqrt{2}$, 2 and $\sqrt{3}$ units are applied to a particle in the directions of vectors (- 1, 0, 1), (1, 0, 1) and (1, 1, - 1) respectively, then find the magnitude and direction of the resultant force.

$$\left[\text{Ans :} \ \sqrt{5}, \ \left(\cos^{-1} \sqrt{\frac{2}{5}}, \ \cos^{-1} \sqrt{\frac{1}{5}}, \ \cos^{-1} \sqrt{\frac{2}{5}} \right) \right]$$

(22) A boat is sailing to the east with a speed of $10\sqrt{2}$ km / hr. A man on boat feels that the wind is blowing from the south-east with a speed of 5 km / hr. Find the true velocity of the wind.

$$\left[\text{Ans :} \ 5\sqrt{5} \ \text{km / hr at an angle} \ \cos^{-1} \frac{3}{\sqrt{10}} \ \text{with east towards north} \right]$$

(23)   A force of magnitude $2\sqrt{10}$ units is acting on a particle in the direction $3\bar{i} - \bar{j}$ and a force of magnitude $3\sqrt{13}$ units is acting on the same particle in the direction $2\bar{i} + 3\bar{j}$. Under the influence of these forces, the particle is displaced from A (1, 2) to B (6, 4). Find the work done.   [Ans: 74 units]

(24)   Prove that the diagonals of a rhombus bisect each other orthogonally.

(25)   If a pair of medians of a triangle are equal, then show that the triangle is isosceles.

(26)   Show that the perpendicular bisectors of sides of any triangle are concurrent.

(27)   Prove that the diagonals of a rhombus are bisectors of its angles.

(28)   If $\overrightarrow{AD}$ is a bisector of $\angle BAC$ in triangle ABC and if D ∈ BC, then show that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.

(29)   ABCDEF is a regular hexagon. Prove that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

(30)   Show that centroid and in-centre of an equilateral triangle are the same. Find the incentre of the triangle with vertices (6, 4, 6), (12, 4, 0) and (4, 2, - 2).

$$\left[\text{Ans} : \left(\frac{22}{3}, \ \frac{10}{3}, \ \frac{4}{3} \right) \right]$$

(31)   If A is (1, 2, 1) and B is (4, - 1, 2), then find S (x, y, z) such that $2\overrightarrow{AB} = \overrightarrow{AS}$.

[Ans: (7, - 4, 3)]

(32)   Let A (1, 2, - 1) and B (3, 2, 2) be given points. Find in which ratios from A and at which points do the XY-, YZ- and ZX-planes divide $\overline{AB}$.

$$\left[\textbf{Ans :}\ \textbf{1 : 2,}\ \left(\frac{5}{3}, 2, 0 \right);\ \textbf{- 1 : 3,}\ \left(0, 2, \frac{5}{2} \right);\ \overleftrightarrow{AB}\ \textbf{is parallel to ZX - plane} \right]$$

(33) Show that (6, 0, 1), (8, - 3, 7) and (2, - 5, 10) can be three vertices of some rhombus. Find the co-ordinates of the fourth vertex of this rhombus.

[**Ans:** (0, - 2, 4)]

(34) Show that (1, 2, 4), (- 1, 1, 1), (6, 3, 8) and (2, 1, 2) are the vertices of a trapezium. Find the area of this trapezium.

$$\left[\textbf{Ans :}\ \frac{3}{2} \sqrt{59} \right]$$

(35) Find the area of the parallelogram ABCD if $\overrightarrow{AC} = \overline{a}$ and $\overrightarrow{BD} = \overline{b}$.

$$\left[\textbf{Ans :}\ \frac{1}{2} \left| \overline{a} \times \overline{b} \right| \right]$$

(36) Find the volume of a prism having a vertex at origin and having edges $\overrightarrow{OA} = \overline{i} + \overline{j} + \overline{k},$ $\overrightarrow{OB} = 3\overline{i} - \overline{j} + \overline{k}$ and $\overrightarrow{OC} = -\overline{i} + \overline{j} - \overline{k}.$

[**Ans:** 4 cubic units]

(37) Show that (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (- 4, 4, 4) cannot be the vertices of any tetrahedron.

(38) Find the volume of the tetrahedron with vertices (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (1, 2, 3).

$$\left[\textbf{Ans :}\ \frac{28}{3}\ \textbf{cubic units} \right]$$

(39) A mechanical boat is rowing towards the north with speed of 8 km / hr. If wind blows from the east with the speed of 10 km / hr, find the resulting speed of the boat and also the direction of resulting motion of the boat.

$$\left[\text{Ans : } 2\sqrt{41} \text{ km / hr at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{41}}\right) \text{ with east towards north} \right]$$

(40) A river flows with a speed of 5 units. A person desires to cross the river in a direction perpendicular to its flow. Find in which direction should he swim if his speed is 8 units.

$$\left[\text{Ans : At an angle of } \pi - \cos^{-1}\left(\frac{5}{8}\right) \text{ with the direction of flow of the river} \right]$$

(41) If speed of a particle is 5 units towards the east and $\sqrt{8}$ units towards the south-west, then find the resultant speed of the particle and its direction.

$$\left[\text{Ans : } \sqrt{13} \text{ units at an angle of } \cos^{-1}\frac{3}{\sqrt{13}} \text{ with east towards south} \right]$$

(42) A boat speeds towards the north at $6\sqrt{2}$ units. A man on the boat feels that the wind is blowing from the south-east at 5 units. Find the true velocity of the wind.

$$\left[\text{Ans : } \sqrt{157} \text{ units at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{314}}\right) \text{ with east towards north} \right]$$

(43) A steamer moves to the north-east with a speed of 40 units. A passenger on the steamer feels the wind to be blowing from the north with $25\sqrt{2}$ units. Find the true velocity of the wind.

$$\left[\text{Ans : } 5\sqrt{34} \text{ units at an angle of } \cos^{-1}\frac{4}{\sqrt{17}} \text{ with east towards south} \right]$$

(44) A particle is displaced from A (2, 1) to B (4, 2) when forces of magnitudes $4\sqrt{5}$ in the direction $2\bar{i} + \bar{j}$ and $6\sqrt{5}$ in the direction $\bar{i} - 2\bar{j}$ are applied. Find the work done.

[Ans: 20 units]