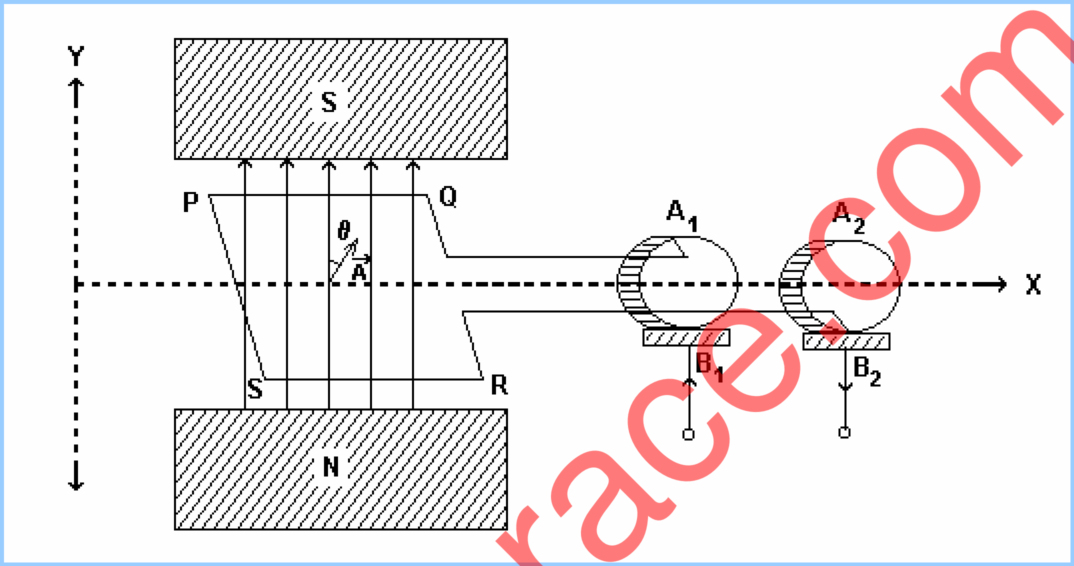
8 - ALTERNATING CURRENT

8.1 Alternating Voltage and Alternating Current (A. C.)

The following figure shows N turns of a coil of conducting wire PQRS rotating with a uniform angular speed ω with respect to the X-axis in a uniform magnetic field along the Y-axis.



Let the angle between the direction of the area vector of the coil and the magnetic field direction be zero at time, t = 0 and $\theta = \omega t$ at time t = t. The magnetic flux, Φ_0, associated with the coil at time t = 0 and Φ_t at time t = t are given by

$$\Phi_0 = N \vec{A} \cdot \vec{B} = N A B \cos 0 = N A B \quad \text{and}$$

$$\Phi_t = N A B \cos \omega t$$

The emf induced in the coil according to Faraday's law is

$$V = - \frac{d \Phi_t}{dt} = N A B \omega \sin \omega t = V_m \sin \omega t \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

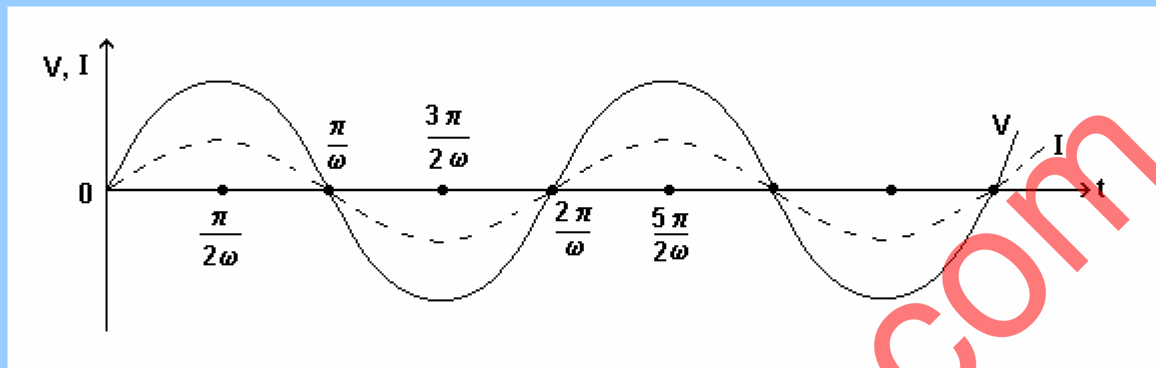
where $V_m = N A B \omega$ is the maximum value of the induced emf.

Equation (1) shows that the induced emf versus time is a sine curve. This emf is obtained between the brushes B_1 and B_2 which are in contact with the slip rings A_1 and A_2 as shown in the above figure.

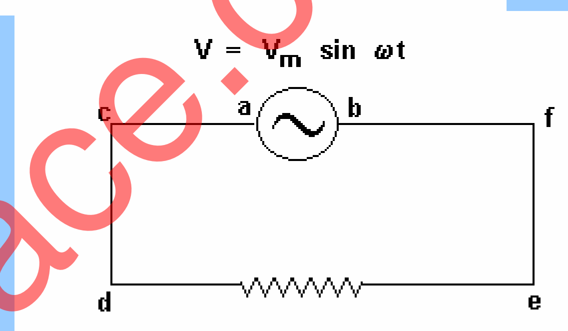
The voltage is zero at time t = 0 and varies as per the function $\sin \omega t$ reaching maximum value V_m at time $t = \pi / 2\omega$ and again zero at time $t = \pi / \omega$. B_2 being at greater potential than B_1 acts like a positive end of the voltage source during this time interval.

After time $t = \pi / \omega$, the potential of B_1 starts to rise with respect to B_2 till time $t = 3 \pi / 2\omega$, reaches maximum in the reverse direction and again becomes zero at time $t = 2\pi / \omega$. This cycle keeps repeating in every time interval of $T = 2\pi / \omega$.

The voltage so developed is known as alternating voltage and its graph versus time is shown by a continuous line in the following figure. The arrangement to produce such a voltage is known as A.C. (alternating current) dynamo or generator.



In the adjoining figure, an alternating voltage source is connected in a circuit with resistor, R. The voltage between a and b is zero at time t = 0. Zero current will flow at this time.



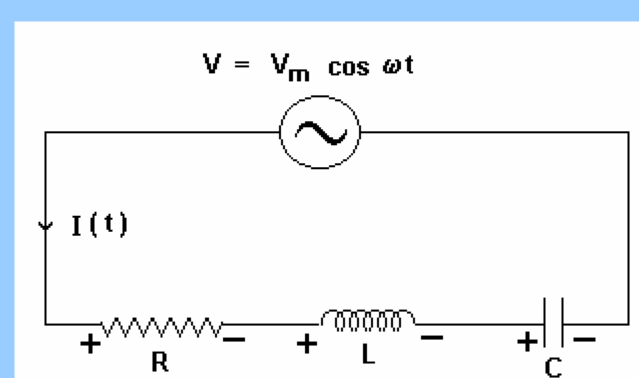
The voltage in the circuit varies according as $V = V_m \sin \omega t$. The current as per Ohm's law will be $I = \frac{V_m \sin \omega t}{R}$. The current changes sinusoidally in the same way as the voltage as shown by the broken line in the above graph.

Here, voltage was considered changing with time as per $\sin \omega t$. Both the voltage and the current can be considered to be changing as per $\cos \omega t$. It is not necessary that the voltage and current should change as above. There are also other ways in which both can change periodically with time.

8.2 A. C. Circuit with Series Combination of Resistor, Inductor and Capacitor

The following figure shows a series combination of resistor having resistance R, inductor of inductance L and a capacitor of capacitance C with an alternating voltage source of voltage changing with time as $V = V_m \cos \omega t$. It is assumed that the resistor has zero inductance and the inductor has zero resistance.

Let I = current in the circuit,
Q = charge on the capacitor and
$\frac{dI}{dt}$ = rate of change of current at time t.



Hence, by Kirchhoff's law,

$$V_R + V_L + V_C = V$$

$$\therefore \; IR + L\frac{dI}{dt} + \frac{Q}{C} = V_m \cos \omega t$$

Putting $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ in the above equation,

$$R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} + \frac{Q}{C} = V_m \cos \omega t$$

$$\therefore \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{V_m}{L}\cos \omega t$$

This equation is known as the differential equation of Q for the series R – L - C A. C. Circuit. It is similar to the differential equation of forced harmonic oscillations given as under.

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}\sin \omega t$$

The mechanical and electrical quantities in the above equations are compared in the following table.

No.	Mechanical quantities	Symbol		Electrical quantities	Symbol
1.	Displacement	y		Electrical charge	Q
2.	Velocity	$\frac{dy}{dt}$		Electric current	$\frac{dQ}{dt} = I$
3.	Co-efficient of resistance	b		Resistance	R
4.	Mass	m		Inductance	L
5.	Force constant	k		Inverse of capacitance	$\frac{1}{C}$
6.	Angular frequency	$\sqrt{\frac{k}{m}}$		Angular frequency	$\frac{1}{\sqrt{LC}}$
7.	Periodic force	F_0		Periodic voltage	V_m

The function of charge Q versus time t which satisfies the above differential equation of charge Q is called its solution. Complex function is used to find the solution of the above differential equation.

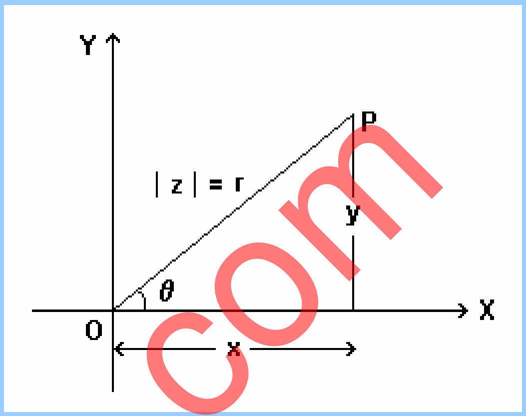
8.3 Complex Numbers (For Information Only)

A complex number $z = x + jy$, where $j = \sqrt{-1}$. x is the real part and y is the imaginary part of the complex number.

(1) The complex number can be represented by a point in a complex plane with x-axis representing real numbers and y-axis representing imaginary numbers. Point P in the figure (next page) represents the complex number $z = x + jy$. The x-coordinate of P gives the real part of z and y-coordinate gives its imaginary part. The magnitude of complex number is equal to r, i.e., $|z| = r = \sqrt{x^2 + y^2}$.

Now, $x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore\ z = r\cos\theta + j\,r\sin\theta$$
$$= r(\cos\theta + j\sin\theta)$$
$$= |z|\,e^{j\theta} \quad (\because\ e^{j\theta} = \cos\theta + j\sin\theta)$$



(2) $z^* = x - jy$ is the complex conjugate of the complex number z.

(3) For complex number z,

$$\frac{1}{z} = \frac{1}{z}\,\frac{z^*}{z^*} = \frac{z^*}{|z|^2} = \frac{x - jy}{x^2 + y^2}$$
$$= \frac{x}{x^2 + y^2} - j\frac{y}{x^2 + y^2}$$

The real part of $\frac{1}{z} = \frac{x}{x^2 + y^2}$ and the imaginary part of $\frac{1}{z} = -j\frac{y}{x^2 + y^2}$.

The real part of the complex number z is denoted by Re (z) and the imaginary part by Im (z).

8.4 Solution of the differential equation

The equation for series L-C-R circuit is $IR + L\frac{dI}{dt} + \frac{Q}{C} = V_m\cos\omega t$

$$\therefore\ IR + L\frac{dI}{dt} + \frac{\int I\,dt}{C} = V_m\cos\omega t \quad \text{where} \quad Q = \int I\,dt$$

$$\therefore\ \frac{dI}{dt} + \frac{R}{L}I + \frac{1}{LC}\int I\,dt = \frac{V_m}{L}\cos\omega t$$

Replacing the current I by complex current i and $\cos\omega t$ by $e^{j\omega t}$, and solving the differential equation of complex quantities, the real part of complex current i will give the equation of real current I as a function of time, t.

Thus, the differential equation of complex quantities is

$$\frac{di}{dt} + \frac{R}{L}i + \frac{1}{LC}\int i\,dt = \frac{V_m}{L}e^{j\omega t} \quad \dots \dots (1),$$ where R, L, C and t are real quantities.

Let $i = i_m e^{j\omega t}$ be a trial solution.

$$\therefore\ \frac{di}{dt} = i_m \cdot j\omega \cdot e^{j\omega t} \quad \text{and} \quad \int I\,dt = \frac{i_m e^{j\omega t}}{j\omega}$$

Putting the values of i, $\frac{di}{dt}$ and $\int I\,dt$ in equation (1) above, we have

$$i_m \cdot j\omega \cdot e^{j\omega t} + \frac{R}{L}i_m e^{j\omega t} + \frac{1}{LC}\frac{i_m e^{j\omega t}}{j\omega} = \frac{V_m}{L}e^{j\omega t}$$

$$\therefore \quad i_m \left(j\omega + \frac{R}{L} + \frac{1}{j\omega LC} \right) = \frac{V_m}{L}$$

$$\therefore \quad i_m \left(j\omega L + R + \frac{1}{j\omega C} \right) = V_m$$

$$\therefore \quad i_m \left(j\omega L + R - \frac{j}{\omega C} \right) = V_m \qquad \left(\because \frac{1}{j} = \frac{j}{j^2} = -j \right)$$

$$\therefore \quad i_m = \frac{V_m}{R + j\left(\omega L - \dfrac{1}{\omega C} \right)}$$

Putting this value of i_m in the trial solution, we have

$$i = \frac{V_m e^{j\omega t}}{R + j\left(\omega L - \dfrac{1}{\omega C} \right)}$$

This equation shows that the resistance offered by an inductor and a capacitor are $j\omega L$ and $-j/\omega C$ which are known as inductive reactance and capacitive reactance respectively. Their magnitudes are ωL and $1/\omega C$ respectively. The inductive and capacitive reactance are represented by symbols Z_L and Z_C while their magnitudes are equal to X_L and X_C.

$Z_L = j\omega L, \qquad Z_C = -j/\omega C,$
$X_L = \omega L, \qquad X_C = 1/\omega C.$

The summation of Z_L, Z_C and R is called the impedance (Z) of the series L-C-R circuit. The unit of impedance is ohm

$$\therefore \quad Z = R + Z_L + Z_C = R + j(\omega L - 1/\omega C)$$

$$\therefore \quad i = \frac{V_m e^{j\omega t}}{Z} = \frac{V}{Z}$$

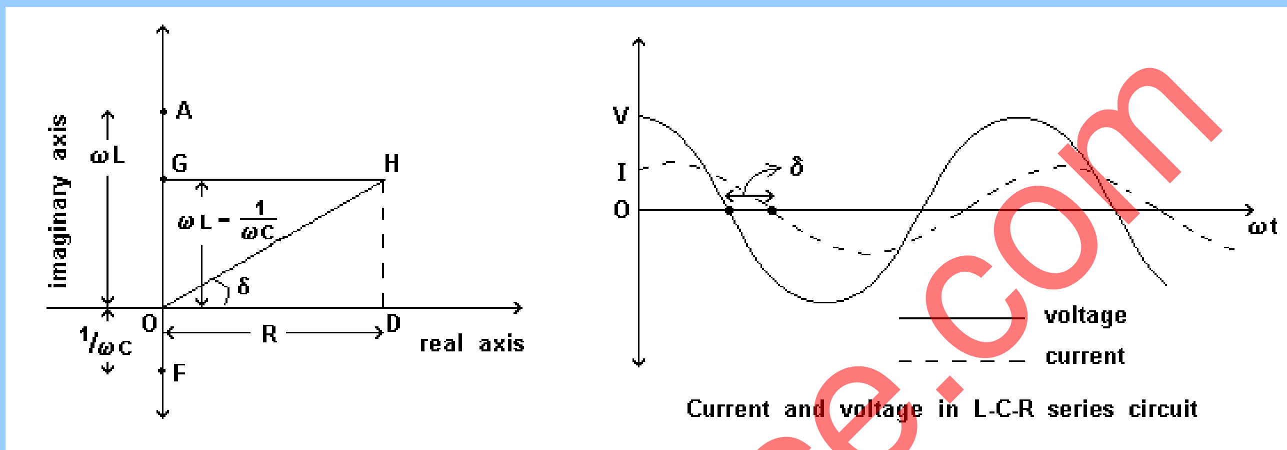
The above equation represents Ohm's law for complex current, complex voltage and impedance. Impedance is also complex which can be expressed as $Z = |Z| e^{j\delta}$.

$$\therefore \quad i = \frac{V_m e^{j\omega t}}{|Z| e^{j\delta}} = \frac{V}{|Z|} e^{j(\omega t - \delta)} = \frac{V}{|Z|} [\cos(\omega t - \delta) + j \sin(\omega t - \delta)]$$

where, $|Z| = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C} \right)^2}$

Now, $I = Re(i) = \dfrac{V_m \cos(\omega t - \delta)}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C} \right)^2}} = \dfrac{V_m \cos(\omega t - \delta)}{|Z|}$

The equation shows that the current in the circuit changes according to cos ($\omega t - \delta$) and lags the voltage by phase δ as shown in the following graph.



Current and voltage in L-C-R series circuit

In the second graph above, point H shows the complex impedance, $Z = R + j(\omega L - 1/\omega C)$ in the complex plane where x-coordinate of the point is R which is the real part of the complex impedance and y-coordinate of the point is $\omega L - 1/\omega C$ which is the imaginary part of the complex impedance.

From the figure,

$$\tan\delta = \frac{HD}{OD} = \frac{\omega L - \frac{1}{\omega C}}{R}$$

and

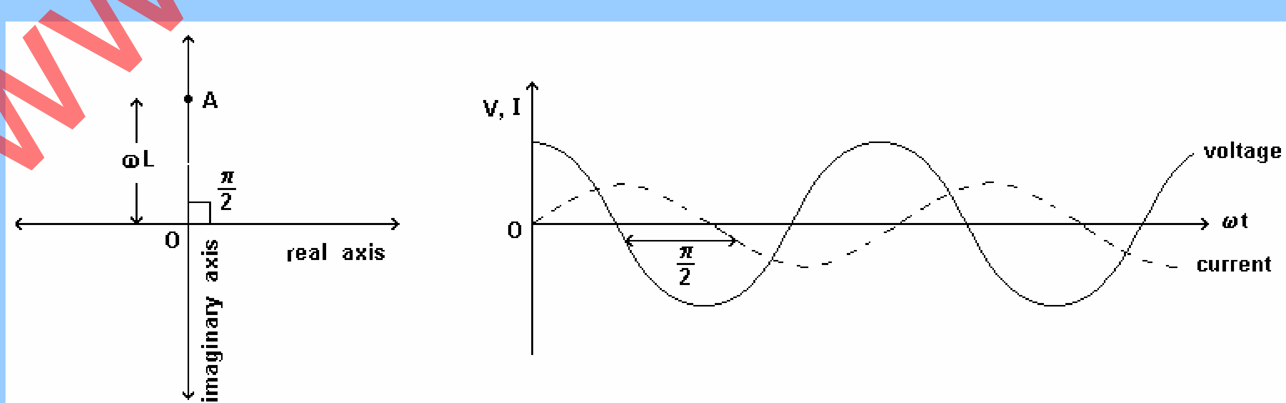
$$|Z| = OH = \sqrt{(OD)^2 + (DH)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

8.5 Different Cases

The rules regarding the equivalent resistances for series and parallel connection of R are also applicable to $j\omega L$ and $-j/\omega C$. Using the above geometrical construction, the relationship between voltage and current for various circuits can be derived.

(1) A. C. Circuit Containing only Inductor:

The impedance $Z = j\omega L = jX_L$ is represented by the point A in the complex plane as shown in the figure.

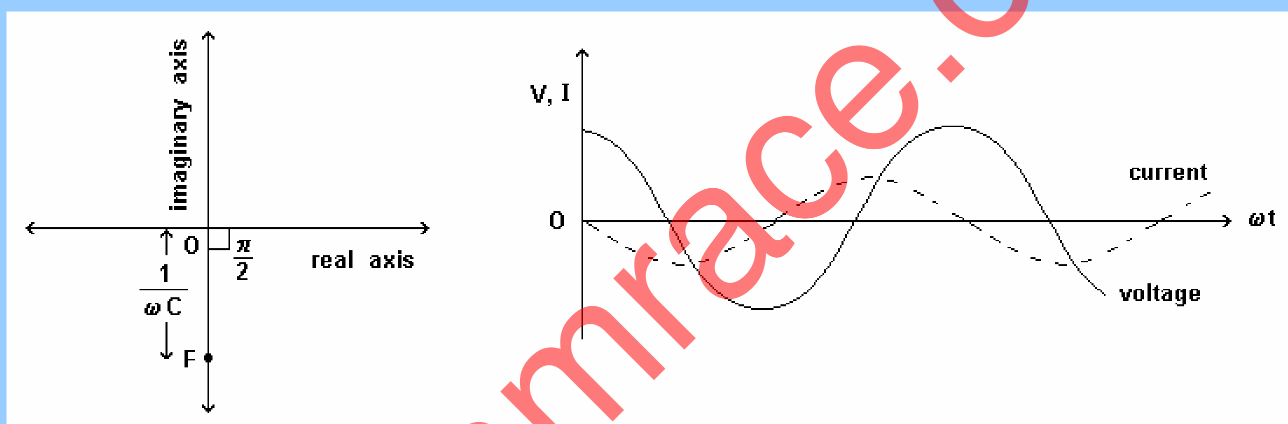


OA forms an angle $\pi / 2$ with the real axis indicating that the current lags the voltage by $\pi / 2$ as shown in the above figures. Further OA = ωL = $|Z|$. So the equation for current can be written as

$$I = \frac{V_m \cos\left(\omega t - \frac{\pi}{2}\right)}{\omega L} = \frac{V_m \cos\left(\omega t - \frac{\pi}{2}\right)}{X_L}$$

(2) A. C. Circuit containing only a capacitor:

The impedance $Z = -j / \omega C = -j X_C$ is represented by the point F in the complex plane as shown in the figure.

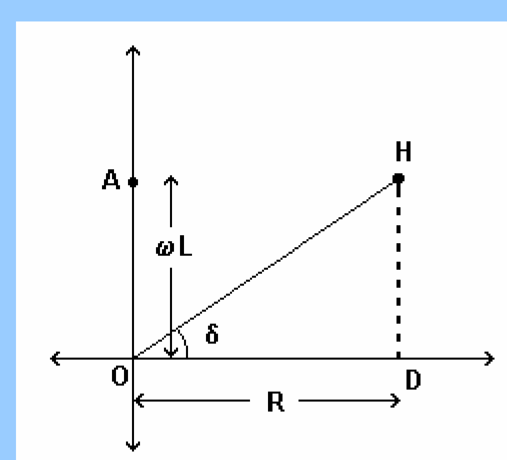


OF forms an angle $-\pi / 2$ with the real axis indicating that the current leads the voltage by $\pi / 2$ as shown in the above figures. Further OF = $1 / \omega C$ = $|Z|$. So the equation for current can be written as

$$I = \frac{V_m \cos\left(\omega t + \frac{\pi}{2}\right)}{\frac{1}{\omega C}} = \frac{V_m \cos\left(\omega t + \frac{\pi}{2}\right)}{X_C}$$

(3) A. C. Circuit containing R and L in series:

The impedance $Z = R + j \omega L$ is represented by the point H in the complex plane as shown in the figure.



$$OH = |Z| = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + X_L^2}$$

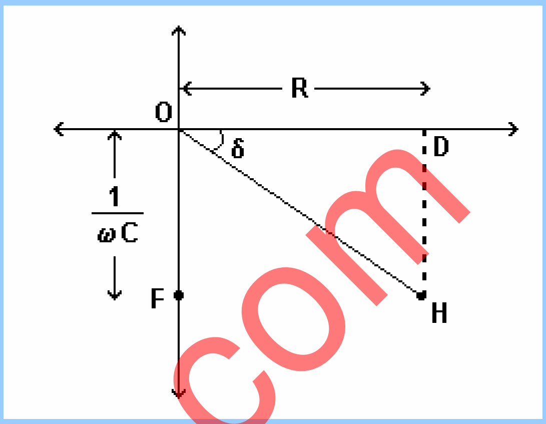
From the figure, $\delta = \tan^{-1}\left[\frac{\omega L}{R}\right] = \tan^{-1}\left[\frac{X_L}{R}\right]$

$$I = \frac{V_m \cos(\omega t - \delta)}{\sqrt{R^2 + \omega^2 L^2}}$$

Here the current lags the voltage by a phase δ, as given by the above equation for δ.

(4) A. C. circuit with R and C connected in series:

The impedance $Z = R - j/\omega C = R - jX_C$ is represented by the point H in the complex plane as shown in the figure.



$$OH = |Z| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} = \sqrt{R^2 + X_C^2}$$

Here, δ is negative and is given by

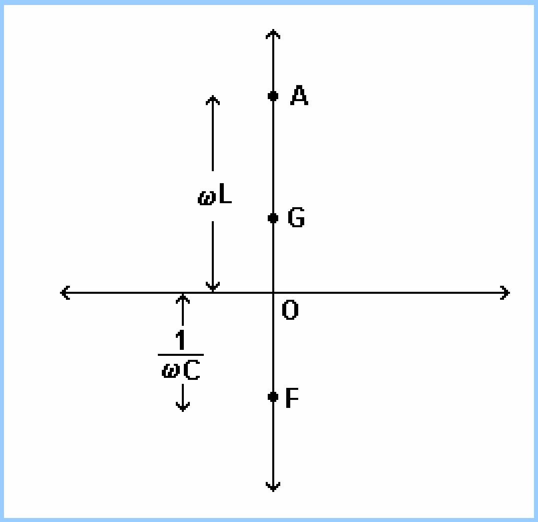
$$\delta = \tan^{-1}\left[\frac{1}{\omega CR}\right] = \tan^{-1}\left[\frac{X_C}{R}\right] \quad \text{and}$$

$$I = \frac{V_m \cos(\omega t + \delta)}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} = \frac{V_m \cos(\omega t + \delta)}{\sqrt{R^2 + X_C^2}}$$

Here the current leads the voltage by a phase δ, as given by the above equation for δ.

(5) An A. C. circuit having L and C connected in series:

The impedance $Z = j\omega L - j/\omega C = jX_L - jX_C$ is represented by the point G in the complex plane as shown in the figure.



$$|Z| = \omega L - 1/\omega C = X_L - X_C$$

Here $\omega L > 1/\omega C$, and current lags voltage by a phase $\pi/2$.

$$\therefore \quad \delta = \pi/2 \quad \text{and} \quad I = \frac{V_m \cos\left(\omega t - \frac{\pi}{2}\right)}{\omega L - \frac{1}{\omega C}}$$

If $\omega L < 1/\omega C$, current leads voltage by a phase $\pi/2$ and

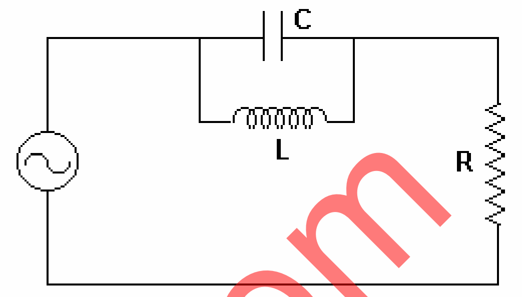
$$\delta = -\pi/2 \quad \text{and} \quad I = \frac{V_m \cos\left(\omega t + \frac{\pi}{2}\right)}{\omega L - \frac{1}{\omega C}}$$

(6) A. C. circuit containing parallel combination of L and C, connected in series with R:

If Z_1 = resultant impedance of the parallel combination of the inductor and the capacitor,

$$\frac{1}{Z_1} = \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{-\frac{j}{\omega C}} + \frac{1}{j\omega L} = j\left[\omega C - \frac{1}{\omega L}\right]$$

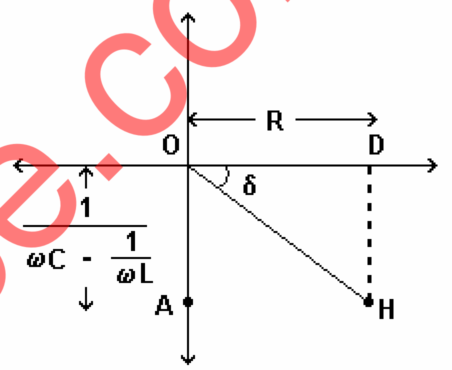
$$\therefore \quad Z_1 = \frac{1}{j\left[\omega C - \frac{1}{\omega L}\right]} = -\frac{j}{\left[\omega C - \frac{1}{\omega L}\right]}$$



$\therefore$ equivalent impedance of the circuit,

$$Z = Z_1 + R = R - \frac{j}{\left[\omega C - \frac{1}{\omega L}\right]}$$

If $\omega C > \frac{1}{\omega L}$, the impedance Z can be represented in the complex plane as shown in the graph.



Here, δ is negative and given by

$$\tan\delta = \frac{HD}{OD} = \frac{1}{R\left[\omega C - \frac{1}{\omega L}\right]}$$

$$|Z| = \sqrt{R^2 + \frac{1}{\left[\omega C - \frac{1}{\omega L}\right]^2}} \quad \text{and} \quad I = \frac{V_m \cos(\omega t + \delta)}{\sqrt{R^2 + \frac{1}{\left[\omega C - \frac{1}{\omega L}\right]^2}}}$$

8.6 r.m.s. Values of Alternating Voltage and Current

Special ammeters and voltmeters are used to measure alternating voltage and current. These meters measure the r.m.s. (root mean square) value of the voltage and current.

Root mean square of a quantity varying periodically with time means the square root of the mean of the squares of the quantity, taken over a time equal to the periodic time.

For $V = V_m \cos \omega t$,

the average value of $V^2 \equiv \langle V^2 \rangle = \langle V_m^2 \cos^2 \omega t \rangle$

$$= V_m^2 \left\langle \frac{1 + \cos 2\omega t}{2} \right\rangle = V_m^2 \left\langle \frac{1}{2} + \frac{\cos 2\omega t}{2} \right\rangle$$

The average value of $1/2$ is $1/2$ and that of $\cos 2\omega t$ is zero for one periodic time.

$$\therefore \langle V^2 \rangle = \frac{V_m^2}{2}$$

$$\therefore \quad V_{r.m.s.} = \sqrt{\langle V^2 \rangle} = \frac{V_m}{\sqrt{2}} \qquad \text{Similarly,} \quad I_{r.m.s.} = \frac{I_m}{\sqrt{2}}$$

8.7 Series Resonance

The current in L – C – R series circuit is given by

$$I = \frac{V_m \cos(\omega t - \delta)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\therefore \quad I = I_m \cos(\omega t - \delta), \qquad \text{where,} \quad I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\text{Now,} \quad I_{r.m.s.} = \frac{I_m}{\sqrt{2}} = \frac{\frac{V_m}{\sqrt{2}}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_{r.m.s.}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_{r.m.s.}}{|Z|}$$

Thus, value of $I_{r.m.s.}$ varies with ω. If $\omega = \omega_0$ is taken such that $\omega_0 L = 1 / \omega_0 C$, then

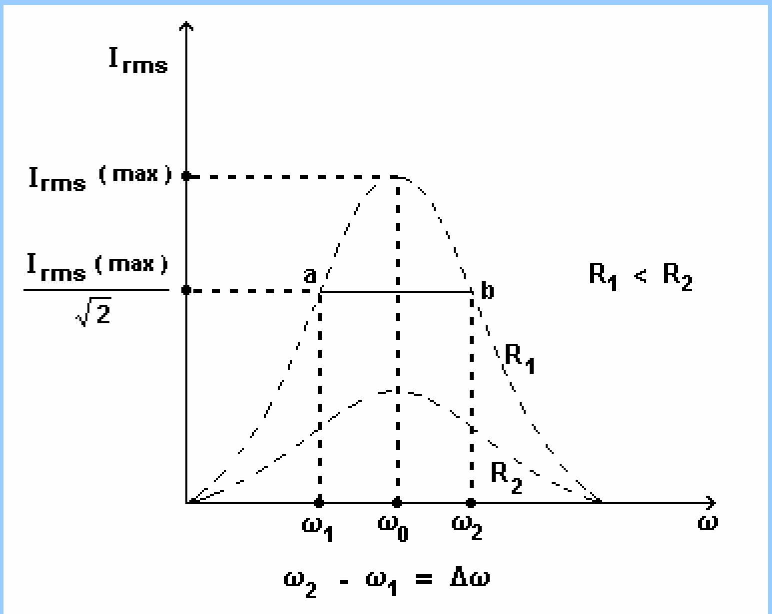
$$I_{r.m.s.} = \frac{V_{r.m.s.}}{R}$$ which is the maximum value of $I_{r.m.s.}$

The L – C – R series circuit is said to be in resonance when the r.m.s. value of the current becomes maximum for a particular frequency, ω_0, of the voltage source.

Now, $\omega_0 = \frac{1}{\sqrt{LC}}$ is known as the natural angular frequency or resonant angular frequency of the given L – C – R series circuit.

Resonance is achieved when the imaginary part of the impedance is zero.

The figure shows the $I_{r.m.s.}$ versus ω curves for two different values of R ($R_1 < R_2$). The resonance curve becomes sharper as the value of R reduces.



Resonance curve

Q - Factor:

The sharpness of the resonance curve of the series L – C – R circuit is measured in terms of a quality known as the Q – factor.

The maximum power $I^2_{r.m.s.} \cdot R$ is obtained during resonance. The power reduces to half of the maximum power when the $I_{r.m.s.}$ is equal to $\frac{I_{r.m.s.(max)}}{\sqrt{2}}$. This situation is shown in the graph where it is found that there are two values of angular frequency, ω_1 and ω_2 for which power is reduced to half the maximum power.

$(\omega_2 - \omega_1) = \Delta\omega$ is known as the half power bandwidth. The sharpness of the resonance increases with the decrease in the value of the half power bandwidth,

The Q – factor is defined as $Q = \frac{\omega_0}{\Delta\omega}$.

Larger the value of Q, sharper will be the resonance curve.

It can be proved that the value of $\Delta\omega = \frac{R}{L}$. Also we know that $\omega_0 = \frac{1}{\sqrt{LC}}$.

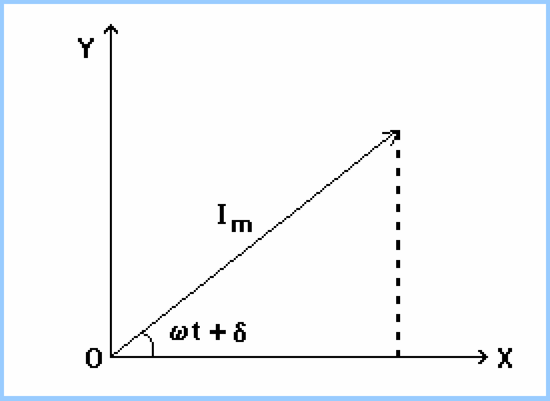
$$\therefore \quad Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Thus, the Q – factor depends on the components of the circuit. It gives the information about the tuning of the circuit as well as the selectivity of the circuit.

To tune a known frequency source, like a T.V. or a radio, one has to select the right value of the inductor or the capacitor. Resonance is obtained only when both L and C are present because at the time of resonance they cancel reactance of each other. Hence, resonance never occurs in case of R-L or R-C circuit.

8.8 Phasor Method

Consider a harmonic function $I = I_m \cos(\omega t + \delta)$.



A vector of magnitude I_m is drawn from the origin of a coordinate system at an angle of $\omega t + \delta$ with the X-axis as shown in the figure.

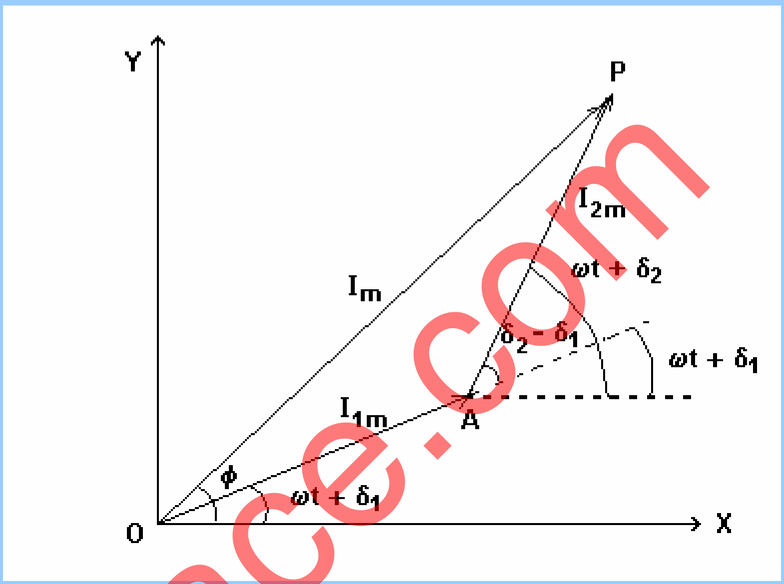
The phase $(\omega t + \delta)$ changes with time, i.e., the angle between the vector shown in the figure and the X-axis keep on changing with time. The vector rotates with the angular frequency ω in the XY plane. Such a rotating vector is known as phasor.

The X-component of the vector at any instant t gives the value of $I_m \cos(\omega t + \delta)$ which is the instantaneous value of the current. To add several functions like $I_1 \cos(\omega t + \delta_1)$, $I_2 \cos(\omega t + \delta_2)$, ... etc., one has to take the algebraic summation of the X-components of the respective phasors. One can deal with the sine function by taking the Y-component of the vector.

Now, consider the example of addition of two harmonic functions,

$I_1 = I_{1m} \cos(\omega t + \delta_1)$ and $I_2 = I_{2m} \cos(\omega t + \delta_2)$.

The figure shows the vectors representing I_1 and I_2 and their resultant vector I . The amplitude of the resultant vector = OP and its phase angle is ϕ using the law of triangle for addition of vectors. The magnitudes of I_1, I_2 and I are given by I_{1m}, I_{2m} and I_m respectively.



The angle between I_1 and I_2 is $\delta_2 - \delta_1$ as can be seen from the figure.

Now,

$$I_m^2 = I_{1m}^2 + I_{2m}^2 + 2 I_{1m} I_{2m} \cos(\delta_2 - \delta_1)$$

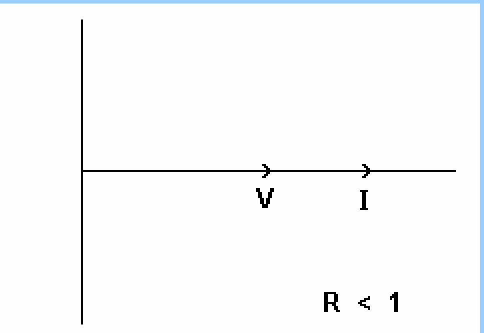
$$= I_{1m}^2 + I_{2m}^2 + 2 I_{1m} I_{2m} \cos \delta$$

where, $\delta = \delta_2 - \delta_1$

8.9 Use of Phasor in an A. C. Circuit

An A. C. Circuit containing only resistor :

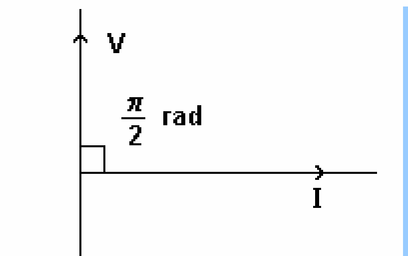
In an A. C. circuit containing only a resistor, the phase difference between the voltage and current is zero ($\delta = 0$). Here, current I is drawn along the X-direction and as $\delta = 0$, voltage is also drawn in the same direction as shown in the figure.



Circuit comprising only resistor

An A. C. Circuit containing only inductor:

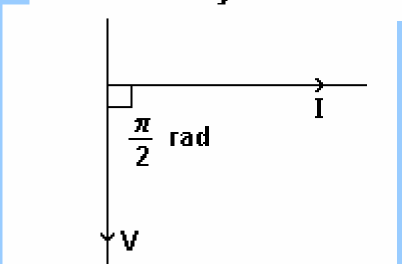
Here, current lags the voltage V by $\pi / 2$ radian, or in other words, the voltage leads the current by $\pi / 2$ radian. Hence, if I is represented along the X-direction, V will be along the Y-direction as shown in the figure.



Circuit comprising only inductor

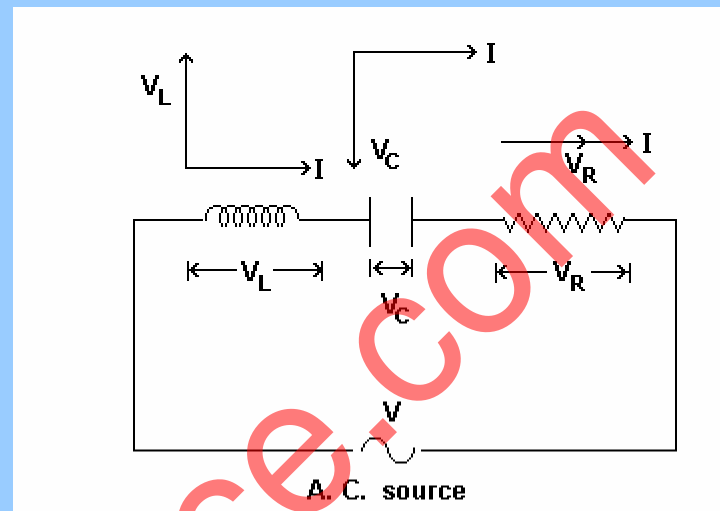
A A. C. circuit containing only capacitor:

Here, current I leads the voltage V by $\pi / 2$ radian, or in other words, the voltage lags the current by $\pi / 2$ radian. Hence, if I is represented along the X-direction, V will be along the negative Y-direction as shown in the figure.



Circuit comprising only capacitor

L - C - R series A. C. circuit:

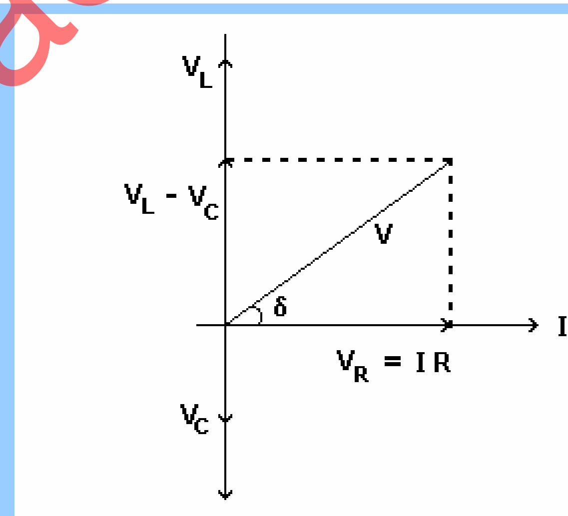


L - C - R series circuit

The phasor diagram of each of the component is shown in the figure. Here, since L, C and R are in series, the same current flows through all the three components. If V is the applied voltage, then

$$V = V_L + V_C + V_R$$

where, V_L, V_C and V_R are the voltage drop across the inductor, the capacitor and the resistor respectively. If the current, I, flowing through the above circuit is represented along the X-direction, then the phasor diagram of the series circuit will be as shown in the next figure.



It can be seen from the figure that

$$V^2 = (V_L - V_C)^2 + V_R^2$$

If I_m is the maximum current, then

$V_R = I_m R$, $V_L = I_m X_L$ and $V_C = I_m X_C$

$$\therefore \; V^2 = I_m^2 (X_L - X_C)^2 + I_m^2 R^2$$

$$\therefore \; V = I_m \sqrt{(X_L - X_C)^2 + R^2}$$

For maximum current, $V = V_m$

$$\therefore \; I_m = \frac{V_m}{\sqrt{(X_L - X_C)^2 + R^2}} = \frac{V_m}{|Z|}$$

If δ = phase angle between the applied voltage and current phasors and $V_L > V_C$, then the curren lags the voltage by a phase δ. If $V_L < V_C$, then current would have led the voltage.

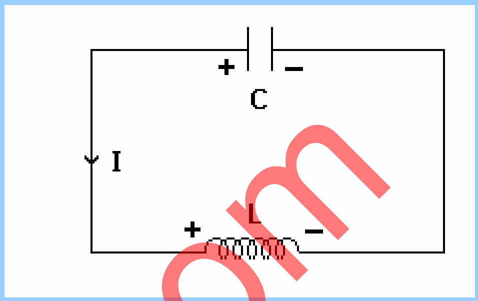
From the above graph,

$$\tan\delta = \frac{V_L - V_C}{V_R} = \frac{I_m X_L - I_m X_C}{I_m R} = \frac{X_L - X_C}{R}$$

8.10 L – C Oscillations

If the two ends of a charged capacitor are connected by a conducting wire or a resistor, then it gets discharged and the energy stored in the form of electric field between its two plates gets dissipated in the form of joule heat.

Now, consider an inductor having negligible resistance connected to a charged capacitor as shown in the figure. Such a circuit is called an L-C circuit.



Let Q_0 be the charge on the capacitor on the initially charged capacitor to which an inductor is connected.

Let Q = charge on the capacitor at time t = t, and
I = circuit current during discharge of capacitor.

∴ applying Kirchhoff's law to the closed circuit, we have

$$\therefore \quad -L\frac{dI}{dt} + \frac{Q}{C} = 0$$

But $I = -\frac{dQ}{dt}$ (negative sign indicates that the charge on the capacitor decreases with time.)

$$\therefore \quad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\therefore \quad \frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$

Comparing the above equation with the differential equation of the simple harmonic motion,

$$\frac{d^2y}{dt^2} = -\omega_0^2 y$$

Q is analogous to the variable y and ω_0^2 is analogous to $\frac{1}{LC}$. The solution to the above equation would be,

$$Q = Q_m \sin(\omega_0 t + \phi) \quad \dots \quad \dots \quad \dots \quad (1)$$

Here, Q_m and ϕ are the constants which can be determined from the initial conditions.

Putting the value of $Q = Q_0$ at time $t = 0$ in the above equation,

$$Q_0 = Q_m \sin\phi \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

Differentiating equation (1) with respect to time t, we have,

$$I = \frac{dQ}{dt} = Q_m \omega_0 \cos(\omega_0 t + \phi)$$

But at time $t = 0$, $I = 0$.

$$\therefore \quad 0 = Q_m \omega_0 \cos\phi \quad \Rightarrow \quad \phi = \pi/2 \quad (\because Q_m \text{ and } \omega_0 \text{ are not zero})$$

Putting the value of $\phi = \pi/2$ in equation (2) gives $Q_m = Q_0$

Putting the value of $\phi = \pi / 2$ and $Q_m = Q_0$ in equation (1), we have,

$$Q = Q_0 \sin (\omega_0 t + \pi / 2) = Q_0 \cos \omega_0 t \quad ... \quad ... \quad ... \quad (3)$$

This equation shows that the charge on the capacitor changes in a periodic manner.

Differentiating the above equation with respect to time t, we have,

$$I = \frac{dQ}{dt} = - Q_0 \omega_0 \sin \omega_0 t \quad ... \quad ... \quad ... \quad (4)$$

This equation shows that the current through the inductor also changes in a periodic manner.

At time $t = 0$, the charge on the capacitor is maximum and the current through the inductor is zero. The electric field intensity and hence the energy associated with the capacitor ($Q^2 / 2C$) is maximum. The energy associated with the magnetic field of the inductor is zero.

With the passage of time, the charge and hence the energy associated with the capacitor decrease as per the equation (3). At the same time the current through the inductor and hence the magnetic field and energy associated with it ($LI^2 / 2$) increase as per the equation (4). It can thus be concluded that the energy of the electric field of the capacitor gets converted into the energy of the magnetic field of the inductor.

At time $t = \pi / (2\omega_0)$, $Q = 0$, and I becomes maximum and the entire energy stored in the electric field gets converted into the energy stored in the magnetic field.

At time $t = \pi / \omega_0$, the charge on the capacitor again becomes maximum but with reverse polarity and the current in the inductor becomes zero. This phenomenon of charge oscillating between the capacitor and inductor in a periodic manner is known as L-C oscillations.

These oscillations of charge results in the emission of electromagnetic radiations which result in the decrease of energy associated with the L-C circuit. Such a circuit is known as the tank circuit of the oscillator.

8.11 Power and Energy associated with L, C and R in an A. C. Circuit

In an A. C. circuit voltage and current continuously change with time. In a series L-C-R circuit, the instantaneous power

$$= VI = V_m \cos \omega t \cdot I_m \cos (\omega t - \delta)$$

$$= V_m I_m \cos \omega t \cdot \cos (\omega t - \delta) = \frac{V_m I_m}{2} [\cos \delta + \cos (2\omega t - \delta)]$$

$\therefore$ Real power, P = Average value of instantaneous power for the entire cycle

$$= \frac{V_m I_m}{2} \left[\frac{1}{T} \int_0^T \cos \delta \, dt + \frac{1}{T} \int_0^T \cos (2\omega t - \delta) \, dt \right]$$

$$= \frac{V_m I_m}{2} \cdot \frac{T}{T} \cos \delta \qquad \left[\because \frac{1}{T} \int_0^T \cos (2\omega t - \delta) \, dt = 0 \right]$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos \delta = V_{r.m.s.} I_{r.m.s.} \cos \delta \quad ... \quad ... \quad ... \quad (1)$$

Special cases:

(i) Circuit containing only resistor:

$\delta = 0 \quad \therefore \quad P = V_{r.m.s.} I_{r.m.s.}$

(ii) Circuit containing only inductor:

$\delta = \pi / 2 \Rightarrow \cos \pi / 2 = 0 \quad \therefore \quad P = 0$

When the current through the inductor increases, the energy from the voltage source gets stored in the magnetic field linked with the inductor and is given back to the circuit when the current through the inductor decreases. Hence power consumed by the circuit is zero. Thus, there is current in an A. C. circuit containing inductor without consuming any power.

(iii) Circuit containing only capacitor:

Here, $\delta = - \pi / 2 \Rightarrow \cos (- \pi / 2) = 0 \quad \therefore \quad P = 0.$

The energy consumed in charging the capacitor is stored in the electric field between the plates of the capacitor and is given back to the circuit when current in the circuit decreases.

(iv) Series L-C-R circuit:

$$\text{Here, } \cos \delta = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)}}$$

Putting this value of $\cos \delta$ in equation (1), we get the value of the power in the series L-C-R circuit. It is less than when only resistance is present in the circuit. Its value is maximum when $\omega^2 LC = 1$.

Thus, power in A. C. circuit containing only inductor or capacitor is zero. The current flowing in such a circuit is called wattless current.

8.12 Transformer

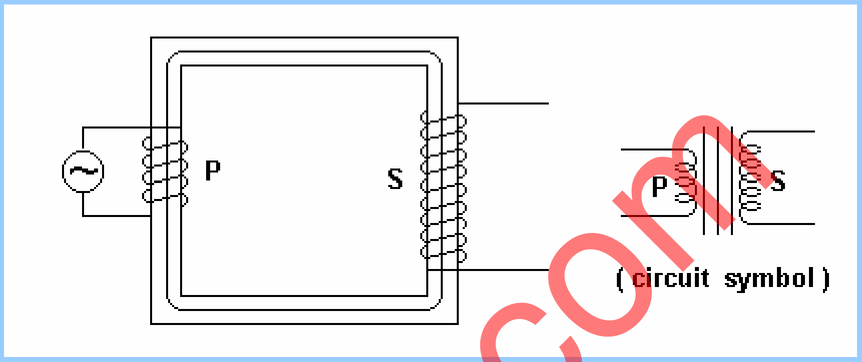
When power $P = V I$ has to be transmitted over a long distance from the power station to the city through the cable having resistance R, I^2R amount of power gets lost in the form of heat which is very large. If this power is transmitted at a very high voltage, then the current I will be less thereby reducing the power loss. Moreover, before supplying this power to the household, its voltage has to be reduced to a proper value.

Both the increase as well as decrease of voltage can be done using a transformer. The transformers, which increase the voltage are called step-up transformers, while those which decrease the voltage are called step-down transformers. There is no appreciable loss of power in the transformers.

Principle:

Transformers work on the principle of electromagnetic induction.

Construction:



The figure shows the construction of a step-up transformer and its circuit symbol.

Two coils of wire are wound very close to each other on the rectangular slab of iron. The rectangular slab of iron is made up of several layers of iron plates to minimize the eddy currents and power loss due to them. One of the coils called primary coil, P, is connected to the A. C. source. The other coil is called the secondary coil, S.

In the step-up transformer shown in the figure, the number of turns of the primary coil is less than that of the secondary coil and is made up of thicker copper wire. In the step-down transformer, the arrangement is opposite to that of a step-up transformer.

The permeability of the material of the slab is high. As a result, all the flux generated by the primary coil remains confined to the core and gets linked to the secondary coil. Therefore, the fluxes, Φ_1 and Φ_2 linked with the primary and the secondary coils are proportional to the respective number of turns N_1 and N_2 of the coils.

$$\therefore \quad \frac{\text{flux } \Phi_2 \text{ linked with secondary coil}}{\text{flux } \Phi_1 \text{ linked with primary coil}} = \frac{\text{No. of secondary turns } N_2}{\text{No. of primary turns } N_1} \quad \dots \quad \dots \quad \dots \quad (1)$$

According to Faraday's law,

the emf induced in the primary coil, $\varepsilon_1 = -\dfrac{d\Phi_1}{dt}$ and

the emf induced in the primary coil, $\varepsilon_2 = -\dfrac{d\Phi_2}{dt}$

From equation (1), we have, $\Phi_2 = \dfrac{N_2}{N_1}\Phi_1 \Rightarrow \dfrac{d\Phi_2}{dt} = \dfrac{N_2}{N_1}\dfrac{d\Phi_1}{dt}$

$$\therefore \quad \varepsilon_2 = \frac{N_2}{N_1}\varepsilon_1 \quad \text{or} \quad \frac{\varepsilon_2}{\varepsilon_1} = \frac{N_2}{N_1} = r$$

r is known as the transformer ratio. For a step-up transformer, $r > 1$.

As the power loss in the transformer is negligible, power input = power output

$$\therefore \quad \varepsilon_2 I_2 = \varepsilon_1 I_1 \quad \Rightarrow \quad \frac{\varepsilon_2}{\varepsilon_1} = \frac{I_2}{I_1} = \frac{N_2}{N_1} = r$$

The above result is valid for an ideal transformer having no power loss. Actually, some power is lost in the primary coil in the form of heat, some in the magnetization and demagnetization of the iron core and some in the form of eddy currents formed on the surface of the iron core. As a result, the output power is less than the input power.