Institute of Actuaries of India

ACET June 2019 Solutions

Mathematics

1. A. $\frac{\log_{16} 625}{\log_8 125} = \frac{\log_{2^4} 5^4}{\log_{2^3} 5^3} = \frac{\log_2 5}{\log_2 5} = 1.$

2. D. In order that $f(x) = \sqrt{\frac{x+2}{x-1}}$ is valid function, $\frac{x+2}{x-1} \geq 0$ and $x \neq 1$.

   These imply:

   Either $x = -2$, or $(x+2)$ and $(x-1)$ must have the same sign.

   If $(x+2) > 0$ and $(x-1) > 0$, then $x > 1$

   Similarly if $(x+2) < 0$ and $(x-1) < 0$, then $x < -2$.

   Putting together, the domain for $f(x)$ is $(-\infty, -2] \cup (1, \infty)$

3. B. $\cos^2 5 + \cos^2 85 = \cos^2 5 + \cos^2(90 - 5) = \cos^2 5 + \sin^2 5 = 1$

   Similarly, $\cos^2 10 + \cos^2 80 = 1, etc.$

   There are 8 such pairs leaving two terms $\cos^2 45 = \frac{1}{2}$ and $\cos^2 90 = 0$.

   Hence, the sum is $8\frac{1}{2}$.

4. B. $f(x) = x^2 - 5; f'(x) = 2x;\ x_0 = 2$.

   Choosing i = 0 in $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_{i+1})}$, we have $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{4} = 2.25.$

5. A. $f(x) = \frac{1}{1+x}$; x real

   $f'(x) = -\frac{1}{(1+x)^2}; f'(0) = -1$

   $f''(x) = \frac{2}{(1+x)^3};\quad f''(0) = 2$

   $f'''(x) = 2\frac{-3}{(1+x)^4}; f'''(0) = 2(-3)$ and so on.

   The Maclaurin's series is: $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

   Hence, $f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

6. C. $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+3)}$ implies

$x^2 + 1 = A(x+3) + B(x-1)(x+3) + C(x-1)^2.$ ... (1)
Letting $x = 1$, in (1) we have, $A = \frac{1}{2}$ and $x = -3$, we have $C = \frac{5}{8}$
Equating the coefficients of x^2: $B + C = 1$ and $B = \frac{3}{8}$. Hence $A + 4B = 2$.
Alternatively, differentiating (1) wrt x, we have $2x = A + 2B(x+1) + 2C(x-1)$ and letting $x = 1$ we have $A + 4B = 2$.

7. D. $\binom{16}{2r} = \binom{16}{3r+1}$ implies $2r = 16 - (3r+1)$ [since $\binom{n}{r} = \binom{n}{n-r}$]

Hence, $r = 3$. Therefore, $\binom{4r}{3} = \binom{12}{3} = 220.$

8. C. The typical term in $\left(2x^2 + \frac{1}{x}\right)^9$ is $\binom{9}{r}(2x^2)^r\left(\frac{1}{x}\right)^{9-r}$.
In order that this term is a constant, we must have $2r = 9 - r$, *i.e.* $3r = 9$ or $r = 3$.
Letting $r = 3$, in the typical term, we have $\binom{9}{3}(2x^2)^3\left(\frac{1}{x}\right)^{9-3} = \binom{9}{3}(2)^3 = 672.$

9. B. $\lim_{x\to 4} \frac{\log_e x - \log_e 4}{x-4}$ is in $\frac{0}{0}$ form .
By L'Hôpital's rule, we have, $\lim_{x\to 4} \frac{\frac{1}{x}}{1} = \frac{1}{4}$.

10. A. $f(x) = \frac{2x}{\log_e x}$, $x > 0$. Hence, $f'(x) = \frac{\log_e x \times 2 - 2x \times \frac{1}{x}}{(\log_e x)^2}$

$= \frac{2(\log_e x - 1)}{(\log_e x)^2} > 0$ if $\log x > 1$ or if $x > e$.

11. B. If $x = r\cos\theta$ then $x^2 = r^2\cos^2\theta$ and $y = r\sin\theta$, then $y^2 = r^2\sin^2\theta$.
$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$.
Hence, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$

12. A. $\int x^3 \log_e x\, dx = \log_e x \cdot \frac{x^4}{4} - \int \frac{x^4}{4}\frac{1}{x}\, dx = \log_e x \cdot \frac{x^4}{4} - \frac{x^4}{16} = \frac{x^4}{4}\left(\log_e x - \frac{1}{4}\right) + c.$

13. C.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, d(\sin x) = \frac{\sin^3 x}{3}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3}.$$

14. C. The function $|x| = -x$ if $x < 0$ and x if $x > 0$. Hence,

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{\infty} \frac{1}{2} e^{-x} dx = \frac{1}{2} + \frac{1}{2} = 1.$$

15. D. Since $\vec{a} \perp \vec{b}$, $\vec{a}.\vec{b} = 0$.

Hence, $(2\vec{\imath} + m\vec{\jmath} + \vec{k}).(\vec{b} = \vec{\imath} - 2\vec{\jmath} + \vec{k}) = 0 \Rightarrow (2)(1) + (m)(-2) + (1)(1) = 0$

$$\Rightarrow 2 - 2m + 1 = 0 \Rightarrow 2m = 3$$

$$\Rightarrow \quad m = \frac{3}{2}.$$

16. B. The projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{2+6-6}{\sqrt{1+4+9}} = \frac{2}{\sqrt{14}}$.

17. D. The rank of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ is 3, since $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 4 \neq 0$

18. A. Given that:

$$A(x) = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[A(x)]^{-1} = \begin{bmatrix} \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} = A(-x).$$

Statistics

19. D. Total number of ways = $\binom{6}{4} + \binom{4}{1}\binom{6}{3} + \binom{4}{2}\binom{6}{2} = 15 + 80 + 90 = 185.$

20. A.

$$P(A \cap B | A \cup B) = \frac{P\big((A \cap B) \cap (A \cup B)\big)}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}.$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.3 - 0.1 = 0.7.$$

So, $P(A \cap B | A \cup B) = \frac{0.1}{0.7} = \frac{1}{7}$

21. C. Suppose Failure is denoted by F.

$P(F|A) = 0.20,\ P(F|B) = 0.10,\ P(A) = 0.70,\ P(B) = 0.30$. So

$$P(A|F) = \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B)} = \frac{0.20 \times 0.70}{0.20 \times 0.70 + 0.1 \times 0.30} = \frac{0.14}{0.17} = \frac{14}{17}.$$

22. C. Mean = 0. Standard deviation cannot be 0. The distribution is symmetric. Mode = 0 (since 0 has highest frequency).

23. B. Median of new observations = $10 + 4 \times 12.8 = 61.2$.

24. A. $P(1 < X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = 0.15 + 0.25 + 0.15 = 0.55.$

25. B. $E(40-2X-X^2) = 40 - 2 \times E(X) - E(X^2)$

$$X \sim \text{Poisson}(3). \; E(X) = 3, Var(X) = 3$$

$$E(X^2) = Var(X) + \left(E(X)\right)^2 = 3 + 9 = 12$$

So, $E(40-2X-X^2) = 40 - 2 \times 3 - 12 = 22.$

26. D. $X \sim \text{Binomial}(10, 0.5)$. $Var(X) = 10 \times 0.5 \times 0.5.$

$$Var(Y) = Var\left(\frac{X}{10}\right) = \frac{Var(X)}{10^2} = \frac{10 \times 0.5 \times 0.5}{10^2} = 0.025.$$

27. A. Find k from $\int_0^\infty f(x)dx = 1$.

$$\int_0^\infty f(x)dx = \int_0^\infty kx^3 \, e^{-x/2} \, dx \; = 2^4 k \int_0^\infty u^3 e^{-u} \, du = 16k \times \Gamma(4) = 16k \times 3!$$
$$= 96k.$$

So $k = \frac{1}{96}$.

28. D. $X \sim N(50, 12^2)$.
$P(X > 50) = P\left(\frac{X-50}{12} > \frac{75-50}{12}\right) = P(Z > 2.083) < P(Z > 1.96) =$
0.025 [where $Z \sim N(0,1)$] .

29. B. $Var(X_1X_2) = E((X_1X_2)^2) - \left(E(X_1X_2)\right)^2 = E(X_1^2X_2^2) - \left(E(X_1X_2)\right)^2$
$= E(X_1^2)E(X_2^2) - \left(E(X_1)\right)^2\left(E(X_2)\right)^2$ (X_1 and X_2 are independently distributed).

$E(X_1) = 0$, $Var(X_1) = 0.5$, $E(X_2) = 1$ and $Var(X_2) = 0.8$.

This implies $E(X_1^2) = Var(X_1) + \left(E(X_1)\right)^2 = 0.5$.

$$E(X_2^2) = Var(X_2) + \left(E(X_2)\right)^2 = 0.8 + 1 = 1.8.$$

Hence $Var(X_1X_2) = 0.5 \times 1.8 - 0 \times 1 = 0.90$.

30. B. $r_{XY} = 0.3$. $Corr(-1.5X, 2Y+3) = \frac{Cov(-1.5X,\; 2Y+3)}{sd(-1.5X) \times sd(2Y+3)} = \frac{-1.5 \times 2 \times Cov(X,Y)}{1.5 \times sd(X) \times 2 \times sd(Y)} =$
$\frac{-cov(X,Y)}{sd(X) \times sd(Y)} = -r_{XY} = -0.3.$

31. B. Two regression lines pass through $(\bar{x}, \bar{y})$.

So $\bar{x} + 3\,\bar{y} = 5$

$4\bar{x} + 3\bar{y} = 8$

This gives $(\bar{x}, \bar{y}) = \left(1, \frac{4}{3}\right)$.

Data Interpretation and Data Visualization

32. C. Median = 50 percentile = 52000.

33. A. First quartile $Q_1 = 28000$. Third quartile $Q_3 = 96000$.

Interquartile range = $Q_3 - Q_1 = 96000 - 28000 = 68000$.

34. D. The percentage of families with income between 52000 and 140000 is 40.

35. B. Sales decrease from 3rd to 4th, 6th to 7th, 8th to 9th and 10th to 11th. So answer is 4.

36. C. Percentage increase:

2nd to 3rd year: $(3/22) \times 100 < 20\%$
4th to 5th year: $(2/21) \times 100 < 20\%$
5th to 6th : $(4/23) \times 100 < 20\%$
7th to 8th : $(3/29) \times 100 < 20\%$
9th to 10th : $(3/28) \times 100 < 20\%$
11th to 12th : $(6/29) \times 100 > 20\%$

37. B. $n(P_1) = 35,\ n(P_2) = 45,\ n(P_1 \cup P_2) = 80 - 15 = 65.$
$n(P_1 \cup P_2) = n(P_1) + n(P_2) - n(P_1 \cap P_2).$

So number of employees of who have opted both P_1 and P_2

$n(P_1 \cap P_2) = 35 + 45 - 65 = 15.$

38. C. The number of employees who have opted only P_1

$= n(P_1) - n(P_1 \cap P_2) = 35 - 15 = 20.$

English

39. B
40. C
41. A
42. C
43. A
44. C
45. C
46. A
47. A
48. A
49. C

50. C
51. D
52. C
53. B
54. D
55. A
56. A
57. C
58. C
59. B
60. C
61. D
62. C

Logical Reasoning

63. D.
64. A.
65. C. The given words can be successive specifications of an address, starting from the Room number and specifying up to the district.
66. B. x weeks x days $= (7x + x)$ days $= 8x$ days.
67. B. P's mother has taken legal steps to allow another person to act on her behalf. Therefore, this is the only choice that indicates that a power of attorney has been established.
68. D. One has to count only the cubes that lie completely inside. They form another cube with sides of length 2 cm. The volume is 8 cm^3, and so there are 8 smaller cubes in it.
69. D. At 4 o'clock the minute hand lags the hour hand by 20 minute spaces. In order that the two hands are in opposite directions, the minute hand has to have a net lead of 30 minute spaces. So there should be a gain of 50 minute spaces. The minute hand gains 55 minute spaces in 60 minutes. Therefore 50 minute spaces are gained in $\frac{60}{55} \times 50 = 54\frac{6}{11}$ minutes.
70. C.

****