Probability

Q 1.
Balls are drawn one-by-one without replacement from a box containing 2 black 4 white and 3 red balls till all the balls are drawn. Find the probability that the balls drawn are in the order 2 black. 4 white and 3 red. **(IIT JEE – 1978 – 3 Marks)**

Q 2.
Six boys and six girls sit in a row randomly. Find the probability that
(i) The six girls sit together
(ii) The boys and girls sit alternately. **(IIT JEE – 1979 – 3 Marks)**

Q 3.
An anti -aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2, and 0.1 respectively. What is the probability that the gun hits the plane? **(IIT JEE – 1981 – 2 Marks)**

Q 4.
A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that both A and B are selected is almost 0.3. Is it possible that the probability of B getting selected is 0.9? **(IIT JEE - 1982 - 2 Marks)**

Q 5.
Cards are drawn one by one at random from a well – shuffled full pack of 52 playing cards until 2 aces are obtained to be drawn, first time. If N is the number of cards required to be drawn, then show that $P_r\{N = n\} = (n - 1)(52 - n)(51 - n)/50 \times 49 \times 17 \times 13$ where $2 \leq n \leq 50$ **(IIT JEE – 1983 – 3 Marks)**

Q 6.
A, B, C are events such that **(IIT JEE – 1983 – 2 Marks)**
P (A) = 0.3, P (B) = 0.4, P (C) = 0.8
P (AB) = 0.008, P (AC) = 0.28, P (ABC) = 0.09
If $P(A \cup B \cup C) \geq 0.75$, then show that P (BC) lies in the interval $0.23 \leq x \leq 0.48$

Q 7.
A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them occurs is 1/3. Find the probability of the occurrence of A.
(IIT JEE – 1984 – 2 Marks)

Q 8.
In a certain city two newspapers A and B are published, it is known that 25% of the city population reads A and 20% reads B while 8% reads both A and B. It is also known that 30% of those who read A but not B look into advertisements and 40% of those who read B but not A look into advertisements while 50% of those who read both A and B look into advertisements. What is the percentage of the population that reads an advertisement? **(IIT JEE – 1984 – 4 Marks)**

Q 9.

In a multiple – choice question there are four alternative answers, of which one or more are correct. A candidate will get marks in the questions only if he ticks the correct answers. The candidate decides to tick the answers at random, if he is allowed up to three chances to answer the questions, find the probability that he will get marks in the questions. **(IIT JEE – 1985 – 5 Marks)**

Q 10.

A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement and are tested till all defective articles are found. What is the probability that the testing procedure ends at the twelfth testing. **(IIT JEE – 1986 – 5 Marks)**

Q 11.

A man takes a step forward with probability 0.4 and backwards with probability 0.6 Find the probability that at the end of eleven steps he is one step away from the starting point.

(IIT JEE – 1987 – 3 Marks)

Q 12.

An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white it is not replaced into the urn. Otherwise it is replaced along with another ball of the same color. The process is repeated. Find the probability that the third ball drawn is black. **(IIT JEE – 1987 - 4 Marks)**

Q 13.

A box contains 2 fifty paisa coins, 5 twenty five paisa coins and a certain fixed number N (≥ 2) of ten and five paisa coins. Five coins are taken out of the box at random. Find the probability that the total value of these 5 coins is less than one rupee and fifty paisa. **(IIT JEE – 1988 – 3 Marks)**

Q 14.

Suppose the probability for A to win game B is 0.4. If A has an option of playing either a "best of 3 games" of a "best of 5 games" match against B, which option should be choose so that the probability of his winning the match is higher? (No game ends in a draw). **(IIT JEE – 1989 – 5 Marks)**

Q 15.

A is set containing n elements. A subset P of A is chosen at random. The set A reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements. **(IIT JEE - 1990 – 5 Marks)**

Q 16.

In a test an examine either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he make a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct given that he copied it, is 1/8. Find the probability that he knew the answer to the question given that he correctly answered it. **(IIT JEE – 1991 – 4 Marks)**

Q 17.

A lot contains 50 defective and 50 non defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as **(IIT JEE – 1992 – Marks)**

A = (the first bulb is defective)

B = (the second bulb is non – defective)

C = (the two bulbs are both defective or both non defective)
Determine whether
(i) A, B, C are pair wise independent
(ii) A, B, C are independent.

Q 18.
Numbers are selected at random, one at a time, from the two digit numbers 00, 01, 02. . . 99 with replacement. An even E occurs if only if the product of the two digits of a selected numbers is 18. If four numbers are selected, find probability that the event E occurs at least 3 times. **(IIT JEE – 1993 – 5 Marks)**

Q 19.
An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4 . . . . 12 are picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8? **(IIT JEE - 1994 – 5 Marks)**

Q 20.
In how many ways three girls and nine boys can be seated in two vans, each having numbers seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats? **(IIT JEE - 1994 – 5 Marks)**

Q 21.
Sixteen players S1, S_2 . . . . S_{16} play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength **(IIT JEE - 1997 C – 5 Marks)**
(a) Find the probability that the player S_1 is among the eight winners.
(b) Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.

Q 22.
If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement determine the probability that the roots of equation $x^2 + px + q = 0$ are real. **(IIT JEE – 1997 – 5 Marks)**

Q 23.
Three players, A, B and C, toss a coin cyclically in that order (that A, B, C, A, B, C, A, B . . .)
Till a head shows. Let be the probability that the coin shows a head. Let α, β and γ be, respectively, the probabilities that A, B and C gets the first head. Prove that $\beta = (1 - p)\,\alpha$. Determine α, β and γ (in terms of p) **(IIT JEE - 1998 – 8 Marks)**

Q 24.
Eight players P_1, P_2 . . . P_8 play a knock – out tournament It is known that whenever the players P_i and P_j play, the player P_i will win if i < j. Assuming that the players are paired at random in each round, what is the probability that the player P_4 reaches the final? **(IIT JEE – 1999 – 10 Marks)**

Q 25.
A coin has probability p of showing head when tossed. It is tossed n times. Let P_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1 = 1$, $p_2 = 1 - p^2$ and $p_n = (1 - p).\ p_{n-1} + p\,(1 - p)\,p_{n-2}$ for all $n \geq 3$. **(IIT JEE 2000 - 5 Marks)**

Q 26.

An urn contains m white and n black balls. A ball is drawn at random and is put back into urn along with k additional balls of the same color as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white? **(IIT JEE - 2001 - 5 Marks)**

Q 27.

An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6, is thrown n times and the list of n numbers showing up is noted. What is the probability that, among the numbers 1, 2, 3, 4, 5, 6, only three numbers appear in this list? **(IIT JEE - 2001 - 5 Marks)**

Q 28.

A box contains N coins, m of which is fair and the rest are biased. The probability of getting a head when a fair coin is tossed is 1/2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. What is the probability that the coin drawn is fair? **(IIT JEE - 2002 - 5 Marks)**

Q 29.

For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the 1^{st} exam is p. If he fails in one of the exams then the probability of passing in the next exam is p/2 otherwise it remains the same. Find the probability that he will qualify. **(IIT JEE - 2003 - 2 Marks)**

Q 30.

A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are 2/3, 1/2 and 1/3 respectively. If A is hit then find the probability that B hits the target and C does not,

(IIT JEE - 2003 - 2 Marks)

Q 31.

A and B are two independent events. C is event in which exactly one of A or B occurs. Prove that
$P(C) \geq P(A \cup B)\, P(\bar{A} \cap \bar{B})$ **(IIT JEE - 2004 - 2 Marks)**

Q 32.

A box contains 12 red and 6 white balls. Balls are drawn from the box one at a time without replacement. If in 6 draws there are at least 4 white balls, find the probability that exactly one white is drawn in the next two draws. (Binomial coefficients can be left as such) **(IIT JEE - 2004 - 4 Marks)**

Q 33.

A person goes to office either by car, scooter, bus or train, the probability of which being 1/7, 3/7, 2/7 and 1/7 respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is 2/9, 1/9, 4/9 and 1/9 respectively. Given that he reached office in time, then what is the probability that he travelled by a car. **(IIT JEE - 2005 - 2 Marks)**

<u>Probability – Solutions</u>

<u>Sol. 1.</u>

To draw 2 black, 4 white and 3 red balls in order is same as arranging two black balls at first 2 places, 4 white at next 4 place, (3^{rd} to 6^{th} place) and 3 red at still next 3 places (7^{th} to 9^{th} place), I.e., B_1 B_2 W_1 W_2 W_3 W_4 R_1 R_2 R_3, which can be done in 2! x 4! x 3! Ways. And total ways of arranging all 2 + 4 + 3 = 9 balls is 9!

∴ Required probability = 2! x 4! x 3!/9! = 1/1260

<u>Sol. 2.</u>

(i) 6 boys and 6 girls sit in a row randomly.

Total ways of their seating = 12!

No. of ways in which all the 6 girls sit together = 6! x 7! (Considering all 6 girls as one person)

∴ Probability of all girls sitting together

= 6! x 7!/12! = 720/12 x 11 x 10 x 9 x 8 = 1/132

(ii) Staring with boy, boys can sit in 6! Ways leaving one place between every two boys and two one a last.

B _ B _ B _ B _ B _ B _

These left over places can be occupied by girls in 6! ways.

∴ If we start, with boys. No. of ways of seating boys and girls alternately = 6! x 6!

In the similar manner, if we start with girl, no. of ways of seating boys and girls alternately

= 6! x 6!

G _ G _ G _ G _ G _ G _

Thus total ways of alternate seating arrangements

= 6! x 6! + 6! x 6!

= 2 x 6! x 6!

∴ Probability of making alternate seating arrangement for 6 boys and 6 girls

= 2 x 6! x 6!/12! = 2 x 720/12 x 11 x 10 x 9 x 8 x 7 = 1/462

Sol. 3.

(a). Let us define the events as:

$E_1 \equiv$ First shot hits the target plane,

$E_2 \equiv$ Second shot hits the target plane,

$E_3 \equiv$ third shot hits the target plane,

$E_4 \equiv$ fourth shot hits the target plane

Then ATQ, $P(E_1) = 0.4$; $P(E_2) = 0.3$;

$P(E_3) = 0.2$; $P(E_4) = 0.1$

$\Rightarrow P(\bar{E}_1) = 1 - 0.4 = 0.6$; $P(\bar{E}_2) = 1 - 0.3 = 0.7$

$P(\bar{E}_3) = 1 - 0.2 = 0.8$; $P(\bar{E}_4) = 1 - 0.1 = 0.9$

(where $\bar{E}_1$ denotes not happening of E_1)

Now the gun hits the plane if at least one of the four shots hit the plane.

Also, P (at least one shot hits the plane).

= 1 - P (none of the shots hits the plane)

$= 1 - P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \bar{E}_4)$

$= 1 - P(\bar{E}_1).\ P(\bar{E}_2).\ P(\bar{E}_3).\ P(\bar{E}_4)$

[Using multiplication thm for independent events] $= 1 - 0.6 \times 0.7 \times 0.8 \times 0.9 = 1 - 0.3024 = 0.6976$

Sol. 4.

Let A denote the event that the candidate A is selected and B the event that B is selected. It is given that

$P(A) = 0.5$ . . . . (1)

$P(A \cap B) \leq 0.3$ . . . . (2)

Now, $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$

Or $0.5 + P(B) - P(A \cap B) \leq 1$ [Using (1)]

Or $P(B) \leq 0.5 + P(A \cap B) \leq 0.5 + 0.3$ [Using (2)]

Or $P(B) \leq 0.8$ $\therefore$ P (B) cannot be 0.9

Sol. 5.

We must have one ace in (n – 1) attempts and one ace in the nth attempt. The probability of drawing one ace in first

(n – 1) attempts is ${}^{4}C_1 \times {}^{48}C_{n-2}/{}^{52}C_{n-1}$ and other one ace in the nth attempt is, ${}^{3}C_1/[52 - (n - 1)] = 3/53 - n$

Hence the required probability,

= 4.48! / (n – 2)! (50 – n)! x (n – 1)! (53 – n)/52! x 3/53 - n

= (n – 1) (52 – n) (51 – n)/50. 49. 17. 13

Sol. 6.

Given that

P (A) = 0.3, P (B) = 0.4, P (C) = 0.8

P (AB) = 0.08, P (AC) = 0.28, P (ABC) = 0.09

$P(A \cup B \cup C) \geq 0.75$

To find P (BC) = x (say)

Now we know,

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

$\Rightarrow P(A \cup B \cup C) = 0.3 + 0.4 + 0.8 - 0.08 - x - 0.28 + 0.09 = 123 - x$

Also we have,

$P(A \cup B \cup C) \geq 0.75$ and $P(A \cup B \cup C) \leq 1$

$\therefore 0.75 \leq P(A \cup B \cup C) \leq 1$

$\Rightarrow 0.75 \leq 1.23 - x \leq 1$

$\Rightarrow 0.23 \leq x \leq 0.48$

Sol. 7.

Given that A and B are independent events

$\therefore P(A \cap B) = P(A).\,P(B)$ .... (1)

Also given that $P(A \cap B) = 1/6$ .... (2)

And $P(\bar{A} \cap \bar{B}) = 1/3$ .... (3)

Also P $(\bar{A} \cap \bar{B}) = 1 - P(\bar{A} \cup \bar{B})$

$\Rightarrow P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) P(\bar{A} \cap \bar{B})$

$\Rightarrow 1/3 = 1$ - P (A) - P (B) + 1/6

$\Rightarrow$ P (A) + P (B) . . . . (4)

From (1) and (2) we get

P (A). P (B) = 1/6 . . . . (5)

Let P (A) = x and P (B) = y then eq' s (4) and (5) become

x + y = 5/6, xy = 1/6

$\Rightarrow$ x - y = $\pm \sqrt{(x + y)^2 - 4xy}$

= $\pm \sqrt{25/36 - 4/6} = \pm 1/6$

$\therefore$ We get $x = 1/2$ and $y = 1/3$

Or $x = 1/3$ and $y = 1/2$

Thus $P(A) = 1/2$ and $P(B) = 1/3$ Or P (A) = 1/3 and P (B) = 1/2 .

Sol. 8.

KEY CONCEPT:

(Total prob. Theorem) If E_1, E_2, E_3 . . . . . . E_n are mutually exclusive and exhaustive events and E is an event which can take place in conjunction with any one of E_1 then

P (E) = P (E_1) P (EI E_1) + P (E_2) P (EI E_2) + ... + P (E_n) P (EI E_n) Let P (A) denote the prob. of people reading newspaper A and P (B) that of people reading newspaper B

Then, P (A) = 25/100 = 0.25

P (B) = 20/100 = 0.20, P (AB) = 8/100 = 0.08

Prob. of people reading the newspaper A but not B = P (AB^c)

= P (A) – P(AB) = 0.25 – 0.08 = 0.17

Similarly,

P (A^cB) = P (B) – P (AB) = 0.20 – 0.08 = 0.12

Let E be the event that a person reads an advertisement.

Therefore

ATQ, P (EI AB^c) = 30/100; P (EI A^c B) = 40/100

P (EI AB) = 50/100

∴ By total prob. theorem (as AB^c, A^c B and AB are mutually exclusive)

P (E) = P (EI AB^c) P (AB^c) + P (EI A^c B) P (A^c B) + P (EI AB). P (AB)

= 30/100 x 0.17 + 40/100 x 0.12 + 50/100 x 0.08

= 0.051 + 0.048 + 0.04

Thus the population that reads an advertisements is 13.9%

Sol. 9.

The total number of ways of ticking the answers in any one attempt = 2^4 - 1 = 15.

The student is taking chance at ticking the correct answer, It is reasonable to assume that in order to derive maximum benefit, the three solutions which he submit must be all different.

∴ n = total no. of ways = ${}^{15}C_3$

m = the no. of ways in which the correct solution is excluded ${}^{14}C_3$

Hence the required probability = 1 - ${}^{14}C_3/{}^{15}C_3$ = 1 – 4/5 = 1/5

ALTERNATE SOLUTION:

The candidate may tick one or more of the alternatives. As each alternative may or may not be chosen, the total numbers of exhaustive possibilities are 2^4 - 1 = 15.

Therefore the prob. that the questions are correctly answered by candidate is 1/15.

As such the candidate may be correct on the first, second or third chance. As these events are mutually exclusive, the total probability will be given by

= 1/15 + 14/15 x 1/14 + 14/15 x 13/14 x 1/13 = 1/15 + 1/15 + 1/15 = 3/15 = 1/5

Thus the probability that the candidate gets marks in the question is 1/5.

Sol. 10.

Let A_1 be the event that the lot contains 2 defective articles and A_2 the event that the lot contains 3 defective articles. Also let A be the event that the testing procedure ends at the twelfth testing. Then according to the question :

P (A_1) = 0.4 and P (A_2) = 0.6

Since $0 < P(A_1) < 1$, $0 < P(A_2) < 1$, and $P(A_1) + P(A_2) = 1$

∴ The events A_1, A_2 form a partition of the sample space. Hence by the theorem of total probability for compound events, we have

NOTE THIS STEP:

$P(A) = P(A_1) P(A|A_1) + P(A_2) P(A|A_2)$ . . . . (1)

Here $P(A|A_1)$ is the probability of the event the testing procedure ends at the twelfth testing when the lot contains 2 defective articles. This is possible when out of 20 articles; first 11 draws must contain 10 non defective and 1 defective articles and 12th draw must give a defective article.

∴ $P(A|A_1) = {}^{18}C_{10} \times {}^{2}C_1 / {}^{20}C_{11} \times 1/9 = 11/190$

Similarly, $P(A|A_2) = {}^{17}C_9 \times {}^{3}C_1 / {}^{20}C_{11} \times 1/9 = 11/228$

Now substituting the values of $P(A|A_1)$ and $P(A|A_2)$ in eq. (1), we get

$P(A) = 0.4 \times 11/190 + 0.6 \times 11/228 = 11/475 + 11/380 = 99/1900$

Sol. 11.

Since the man is one step away from starting point means that either

(i) man has taken 6 steps forward and 5 steps backward

Or (ii) man has taken 5 steps forward and 6 steps backward.

Taking movement 1 step forward as success and 1 step backward as failure.

∴ p = Probability of success = 0.4 and q = probability of failure = 0.6

∴ required probability = $P(X = 6 \text{ or } X = 5)$

$= P(X = 6) + P(X = 5)$

$= {}^{11}C_6\, p^6 q^5 + {}^{11}C_5\, p^5 q^6$

$= {}^{11}C_5\, (p^6 q^5 + p^5 q^6) = {}^{11}C_5\, (p + q)(p^5 q^5)$

$= 11.\,10.\,9.\,8.\,7/1.\,2.\,3.\,4.\,5\ (0.4 + 0.6)(0.4 \times 0.6)^5$

$= 462 \times 1 \times (0.24)^5 = 0.37$

Hence the required prob. = 0.37

Sol. 12.

(a) There are following four possible ways of drawing first two balls.

(i) Both the first and the second balls drawn are white.

(ii) The first ball drawn is white and the second ball drawn is black.

(iii) The first ball is black and the second ball drawn is white.

(iv) Both the first and the second balls drawn are black. Let us define events (i), (ii), (iii) and (iv) by E_1, E_2, E_3 and E_4 respectively. Also let E denotes the event that the third ball drawn is black.

Then, $P(E_1) = 2/4 \times 1/3 = 1/6$, $P(E_2) = 2/4 \times 2/3 = 1/3$

$P(E_2) = 2/4 \times 2/5 = 1/5$, $P(E_4) = 2/4 \times 3/5 = 3/10$

Also $P(E \mid E_1) = 1$, since when the event E, has already happened i.e. the first two balls drawn are both white, they are not replaced and so there are left 2 black balls in the urn so that the probability that the third ball drawn in this case is black = 2/2 = 1.

Again $P(E \mid E_2) = 3/4$, since when the event E_2 has already happened there are 3 black and one white balls in the urn. So in this case the probability that the third ball drawn is black = ¾.

Similarly, $P(E \mid E_3) = 3/4$ and $P(E \mid E_4) = 2/3$

Now by thm of total prob. for compound events, we have

$P(E) = P(E_1) P(E \mid E_1) + P(E_2) P(E \mid E_2) + P(E_3) P(E \mid E_3) + P(E_4) P(E \mid E_4)$

$= 1/6 \times 1 + 1/3 \times 3/4 + 1/5 \times 3/4 + 3/10 \times 2/3 = 1/6 + 1/4 + 3/20 + 1/5 = 23/30$

Sol. 13.

Here the total number of coins is N + 7. Therefore the total number of ways of choosing 5 coins out of N + 7 is ${}^{N+7}C_5$ Let E denotes the event that the sum of the values of the coins is less than one rupee and fifty paisa.

Then E′ denotes the event that the total value of the five coins is equal to or more than one rupee and fifty paisa.

NOTE THIS STEP:

The number of cases favorable to E′ is

$= {}^2C_1 \times {}^5C_4 \times {}^NC_0 + {}^2C_2 \times {}^5C_3 \times {}^NC_0 + {}^2C_2 \times {}^5C_2 \times {}^NC_1$

$= 2 \times 5 + 10 + 10N = 10(N + 2)$

∴ P (E) = 10 (N + 2)/$^{n+1}C_5$

⇒ P (E) = 1 – P (E) = 1 – 10 (N + 2)/$^{N+7}C_5$

Sol. 14.

The probability p_1 (say) of winning the best of three games is = the prob. of winning two games + the prob. of winning three games.

= $^3C_2\,(0.6)\,(0.4)^2 + {}^3C_3\,(0.4)^3$ [Using Binomial distribution]

Similarly the probability of winning the best five games is p_2 (say) = the prob. of winning three games + the prob. of winning 5 games.

= $^5C_3\,(0.6)^2\,(0.4)^3 + {}^5C_3\,(0.6)\,(0.4)^3 + {}^5C_5\,(0.4)^5$

We have p_1 = 0.288 + 0.064 = 0.352

And p_2 = 0.2304 + 0.0768 + 0.01024 = 0.31744

As $p_1 > p_2$

∴ A must choose the first offer i.e. best of three games.

Sol. 15.

Set A has a elements.

∴ Number of subsets of A = 2”

∴ Each one of P and Q can be selected in 2” ways.

Hence total no. of ways of selecting P and Q = 2” = 4”.

Let P contains r elements, where r varies from 0 to n, Then, P can be chosen in ^nC_r ways.

Now as P ∩ Q = ϕ, Q can be chosen from the set of all subsets of set consisting of remaining (n – r) elements. This can be done in 2^{n-r} ways.

∴ P and Q can be chosen in $^nC_r \cdot 2^{n-r}$ ways. But, r can vary from 0 to n

∴ total number of disjoint sets P and Q are

$$= \sum_{r=0}^{n} {}^nC_r\, 2^{n-r} = (1+2)^n = 3^n$$

NOTE THIS STEP:

∴ Required probability = $3^n/4^n = (3/4)^n$

ALTERNATE SOLUTION:

Let A = {a_1, a_2, a_3, ..., a_n}

For each a_i, $1 \leq i$ n, there Aries 4 cases

(i) $a_1 \in P$ and $a_1 \in Q$

(ii) $a_1 \notin P$ and $a_1 \in Q$

(iii) $a_1 \in P$ and $a_1 \notin Q$

(iv) $a_1 \notin P$ and $a_1 \notin Q$

∴ total no. of ways of choosing P and Q is 4^n. Here case (i) is not favorable as $P \cap Q = \phi$

∴ For each element there are 3 favorable cases and hence total no. of favorable cases 3"

Hence prob. $(P \cap Q) = \phi) = 3^n/4^n = (3/4)^n$

ALTERNATE SOLUTION:

The set P be the empty set, or one element set or two elements set . . . . or n elements set. Then the set Q will be chosen from amongst the remaining n elements or (n – 1). Element for (n – 2) elements . . . . or no elements. Now if P is the empty set then prob. of its choosing is ${}^nC_0/2^n$, if it is one element set the then prob. of its choosing is ${}^nC_1/2^n$, and so on. When the set P consisting of r elements is chosen from A, then the prob. of choosing the set Q from amongst the remaining n - r elements $2^{n-r}/2^n$. Hence the prob. that P and Q have no common elements is given by

$$\sum_{r=0}^{n} {}^nC_r/2^n \,.\, 2^{n-r}/2^n = 1/4^n \sum_{r=0}^{n} {}^nC_r\, 2^{n-r}$$

$= 1/4^n (1 + 2)^n$ (Using Binomial thm.) $= 3^n/4^n = (3/4)^n$

<u>**Sol. 16.**</u>

KEY CONCEPT :

Baye's theorem: $E_1, E_2, E_3, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events and E is an event which takes place in conjunction with any one of E_1 then the probability of the event E_1 happening when the event E has taken place is given by

$$P(E_1 | E) = \frac{P(E_1)P(E|E_1)}{\sum_{i=1}^{n} P(E_1)P(E|E_1)}$$

Let us define the events :

$A_1 \equiv$ the examinee guesses the answer,

$A_2 \equiv$ the examinee copies the answer

$A_3 \equiv$ the examinee knows the answer,

$A \equiv$ the examinee answers correctly

ATQ, $P(A_1) = 1/3$; $P(A_2) = 1/6$

As any one happens out of A_1, A_2, A_3, these are mutually exclusive and exhaustive events.

$\therefore P(A_1) + P(A_2) + P(A_3) = 1$

$\Rightarrow P(A_3) = 1 - 1/3 - 1/6 = 6 - 2 - 1/6 = 3/6 = 1/2$

Also we have, $P(A| A_1) = 1/4$

[$\because$ out of 4 choices only one is correct.] $P(A| A_2) = 1/8$

(given) $P(A| A_3) = 1$

[If examinee knows the ans., it is correct. i.e. true event]

To find $P(A_3| A)$. By Baye's thm. $P(A_3| A)$

$= P(A_3| A)\ P(A_3)/P(A| A_1)\ P(A_1) + P(A| A_2)\ P(A_2) + P(A| A_3)\ P(A_3)$

$= \frac{1\frac{1}{2}}{\frac{1}{4}.\frac{1}{3}+\frac{1}{8}.\frac{1}{6}+\frac{1.1}{2}} = 1/2/29/48 = 1/2 \text{ x } 18/29 = 24/29.$

Sol. 17.

Let S = defective and Y = non defective. Then all possible outcomes are {XX, XY, YX, YY}

Also P (XX) = 50/100 x 50/100 = 1/4,

P (XY) = 50/100 x 50/100 = 1/4,

P (YX) = 50/100 x 50/100 = 1/4,

P (YY) = 50/100 x 50/100 = 1/4

Here, A =XX ∪ XY; B = XY ∪ YY; C = XX ∪ YY

$\therefore$ P (A) = P (XX) + P (XY) = 1/4 + 1/4 = 1/2

$\therefore$ P (B) = P (XY) + P (YX) = 1/4 + 1/4 = 1/2

P (C) = P (XX) + P (YY) = 1/4 + 1/4 =1/2

Now, P (AB) = P (XY) = 1/4 = P (A). P (B)
∴ A and B are independent events.

P (BC) = P (YX) = 1/4 = P (B). P (C)

∴ B and C are independent events.

P (CA) = P (XX) = 1/4 = P (C). P (A)

∴ C and A are independent events.

P (ABC) = 0 (impossible event)

≠ P (A) P (B) P (C)

∴ A, B, C are dependent events,

Thus we can conclude that A, B, C are pair wise independent bet A, B, C are dependent events.

Sol. 18.

The given numbers are 00, 01, 02 . . . . 99. These are total 100 numbers, out of which the numbers, the product of whose digits is 18, are 29, 36, 63 and 92.

∴ p = P (E) = 4/100 = 1/25 ⇒ q = 1 – p = 24/25

From Binomial distribution

P (E occurring at least 3 times) = P (E occurring 3 times) + P (E occurring 4 times)

$^4C_3\, p^3\, q + {}^4C_4\, p^4 = 4 \times (1/25)^3\, (24/25) + (1/25)^4 = 97/(25)^4$

Sol. 19.

E_1 ≡ number noted is 7, E_2 ≡ number notes is 8,

H ≡ getting head on coin, T ≡ getting tail on coin.

Then by total probability theorem,

$P(E_1) = P(H)\, P(E_1 | H) + P(T)\, P(E_1 | T)$

And $P(E_2) = P(H)\, P(E_2 | H) + P(T)\, P(E_2 | T)$

Where P (H) = 1/2 ; P (T) = 1/2

$P(E_1 | H)$ = prob. of getting a sum of 7 on two dice. Here favorable cases are

{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

$\therefore$ P (E_1| H) = 6/36 = 1/6

Also P (E_1| T) = prob. if getting '7' numbered card out of 11 cards = 1/11.

P (E_2| H) = Prob. of getting a sum of 8 on two dice. Here favorable cases are

{(2, 6), (6, 2), (4, 4), (5, 3), (3, 5)}

$\therefore$ P (E_2| H) = 5/36

P (E_2| T) = prob. of getting '8' numbered card out of 11 cards = 1/11

$\therefore$ P (E_1) = 1/2 x 1/6 + 1/2 x 1/11 = 1/12 + 1/22 = 11 + 6/132 = 17/132

P (E_2) = 1/2 x 5/36 + 1/2 x 1/11 = 1/2 [55 + 36/396] = 91/792

Now E_1 and E_2 are mutually exclusive events therefore

P (E_1 or E_2) = P (E_1) + P (E_2) = 17/132 + 91/792

= 102 + 91/792 = 193/792 = 0.2436

<u>Sol. 20.</u>

We have 14 seats in two vans. And there are 9 boys and 3 girls. The no. of ways of arranging 12 people on 14 seats without restriction is

$^{14}P_{12}$ = 14!/2! = 7 (13!)

Now the no. of ways of choosing back seats is 2. And the no. of ways of arranging 3 girls on adjacent seats is 2 (3!). and the no. of ways of arranging 9 boys on the remaining 11 seats is $^{11}P_9$

Therefore, the required number of ways

= 2. (2.3!). $^{11}P_9$ = 4.3!. 11! /2! = 12!

Hence, the probability of the required event

= 12! /7.13! = 1/91

<u>Sol. 21.</u>

(a) Prob. of S_1 to among the eight winners = (prob. of S_1 being in a pair) x (prob. of S_1 winning in the group).

= 1 x 1/2 [$\because$ S_1 is definitely in a group i.e. certain event] = 1/2

(b) If S_1 and S_2 are in the same pair then exactly on wins. If S_1 and S_2 are in two pairs separately then exactly one of S_1 and S_2 will be among the eight winners if S_1 win and S_2 loses or S_1 loses and S_2 wins.

Now the prob. of S_1, S_2 being in the same pair and one wins.

= (prob. of S_1, S_2 being in the same pair) x (prob. of any one winning in the pair)

And the prob. of S_1, S_2 being in the same pair

= n (E)/n (S), where n (S) = the no. of ways in which 16 person can be divided in 8 pairs; n (E) = the no. of ways in which S_1, S_2 are in same pair or 14 persons can be divided into 7 pairs.

$\therefore$ n (E) = $14!/(2!)^7 .7!$ and n (S) = $16!/(2!)^8 .8!$

$\therefore$ Prob. of S_1 and S_2 being in the same pair

$= \frac{\frac{14!}{(2!)^7 .7!}}{\frac{16!}{(2!)^8 .8!}} = 21.8/16.15 = 1/15$

The prob. of any one winning in the pair of S_1, S_2 = P (certain event) = 1

$\therefore$ The pair of S_1, S_2 being in two pairs separately and any one of S_1, S_2 wins.

= the prob. of S_1, S_2 being in two pairs separately and S_1 wins, S_2 loses + the prob. of S_1, S_2 being in two pairs separately and S_1 loses, S_2 wins.

$$= \left[1 - \frac{\frac{14!}{(2!)^7 .7.!}}{\frac{16!}{(2!)^8 .8!}}\right] \text{x } 1/2 \text{ x } 1/2 + \left[1 - \frac{\frac{14!}{(2!)^7 .7.!}}{\frac{16!}{(2!)^8 .8!}}\right] \text{x } 1/2 \text{ x } 1/2$$

$$= 2x \frac{\frac{16 - 14! x 16}{(2!)^8 .8!}}{\frac{16!}{(2!)^8 .8!}} \text{ x } 1/4 = 1/2 \text{ x } 14 \text{ x } 14!/15 \text{ x } 14! = 7/15$$

$\therefore$ Required prob. = 1/15 + 7/15 = 8/15

<u>**Sol. 22.**</u>

The required probability = 1 – (Probability of the event that the roots of $x^2 + px + q = 0$ are non – real if and only if

$p^2 - 4q < 0$ i.e. if $p^2 < 4q$.

We enumerate the possible values of p and q, for which this can happen in the following table.

q	P	Number of pairs p, q
1	1	1
2	1, 2,	2

3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
10	1, 2, 3, 4, 5, 6	6

Thus, the number of possible pairs = 38. Also, the total number of possible pairs is 10 x 10 = 100.

$\therefore$ The required probability

$= 1 - 38/ 100 = 1 - 0.38 = 0.62$

Sol. 23.

Given that p is the prob. that coin shows a head then $1 - p$ will be the prob. that coin shows a tail.

Now $\alpha = P$ (A gets the 1st head in 1st try)

$\Rightarrow \alpha = P(H) + P(T)P(T)P(T)P(H) + P(T)P(T)P(T)P(T)P(T)P(T)P(H)$

$= p + (1 - p)^3 p + (1 - p)^6 p + \ldots\ldots\ldots\ldots$

$= p [1 + (1 - p)^3 (1 - p)^6 + \ldots\ldots\ldots\ldots]$

$= p/1 - (1 - p)^3$

NOTE THIS STEP . . . (i)

Similarly $\beta = P$ (B gets the 1st head in 1st try) + P (B gets the 1st head in 2nd try) + ...

$= P(T)P(H) + p(T)p(T)p(T)p(T)p(T)P(H) + \ldots\ldots\ldots\ldots$

$= (1 - p)p + (1 - p)^4 + \ldots\ldots\ldots\ldots$

$= (1 - p)p/1 - (1 - p)^3$ . . . . (ii)

From (i) and (ii) we get $\beta = (1 - p)\alpha$

Also (i) and (ii) give expression for α and β in terms of p.

Also $\alpha + \beta + \gamma = 1$(exhaustive events and mutually exclusive events)

$\Rightarrow \gamma = 1 - \alpha - \beta = 1 - \alpha - (1 - p)\alpha$

$= 1 - (2 - p)\alpha = 1 - (2 - p)p/1 - (1 - p)^3$

$= 1 - (1 - p)^3 - (2p - p^2)/1 - (1 - p)^3$

$= 1 - 1 + p^3 + 3p(1 - p) - 2p + p^2/1 - (1 - p)^3$

$= p^3 – 2p^2 + p/1 – (1 – p)^3 = p (p^2 – 2p + 1)/1 – (1 – p)^3 = p (1 – p)^2/1 – (1 – p)^3$

Sol. 24.

The number of ways in which $P_1, P_2 \ldots\ldots\ldots\ldots, P_8$ can be paired in four pairs.

$= 1/4! \times {}^8C_2 \times {}^6C_2 \times {}^8C_2 \times {}^4C_2 \times {}^2C_2 = 105$

Now, at least two players certainly reach the second round in between P_1, P_2, and P_3 and P_4 can reach in final if exactly two players play against each other in between P_1, P_2, P_3 and remaining player will play against one of the players from P_5, P_6, P_7, P_8 and P_4 plays against one of the remaining three from P_5, P_6, P_7, P_8

This can be possible in ${}^3C_2 \times {}^{4C1} \times {}^3C_1 = 36$ ways

∴ Prob. that P_4 and exactly one of $P_5 \ldots\ldots\ldots P_8$ reach second round

$= 36/105 = 12/35$

If P_1, P_i, P_4 and P_j where i = 2 or 3 and j = 5 or 6 or 7 reach the second round then they can be paired in 2 pairs in

$1/2! \times {}^4C_2 \times {}^2C_2 = 3$ ways

But P_4 will reach the final if P_1 plays against P_i and P_4 plays against P_j

Hence the prob. that P_4 reach the final round from the second = 1/3.

∴ prob. that P_4 reach the final is 12/35 x 1/3 = 4/35.

Sol. 25.

Given that the probability of showing head by a coin when tossed = p

∴ Prob. of coin showing a tail = 1 – p

Now p_n = prob. that no two or more consecutive heads occur when tossed n times.

∴ p_1 = prob. of getting one or more on no head = prob. of H or T = 1

Also p_2 = prob. of getting one H or no H

= P (HT) + P (TH) + P (TT)

$= p (1 – p) + p(1 – p) p + (1 – p) (1 – p)$

$= 1 – p^2$, For $n \geq 3$

P_n = prob. that no two or more consecutive heads occur when tossed n times.

= p (last outcome is T) P (no two or more consecutive heads in (n – 1) throw) + P (last outcome H) P ((n – 1)th throw results in a T) P (no two or more consecutive heads in (n – 2) n throws)
= (1 – p) P_{n-1} + p (1 – p) p_{n-2} Hence Proved.

Sol. 26.

Let W_1 (B_1) be the event that a white (a black) ball is drawn in the first draw and let W be the event that a white ball is drawn in the second draw. Then

P (W) = P (B_1). P (W| B_1) + P (W_1). P (W| W_1)

= n/m + n. m/m + n + k + m/m + n. m + k/m + n + k

= m(n + m + k)/(m + n) (m + n + k) = m/m + n

Sol. 27.

The total no. of outcomes = 6^n

We can choose three numbers out of 6 in ${}^6C_3$ ways. By using three numbers out of 6 we can get 3^n sequences of length n. But these sequences of length n which use exactly two numbers and exactly one number.

The number of n – sequences which use exactly two numbers

= ${}^3C_2$ [2^n – 1^n – 1^n] = 3(2^n – 2) and the number of n sequence which are exactly one number

= (${}^3C_1$) (I^n) = 3

Thus, the number of sequences, which use exactly three numbers

= ${}^6C_3$ [3^n – 3(2^n – 2) – 3] = ${}^6C_3$ [3^n – 3(2^n) + 3]

∴ Probability of the required event,

= ${}^6C_3$ [3^n – 3(2^n) + 3]/6^n

Sol. 28.

Let E_1 be the event that the coin drawn is fair and E_2 be the event that the coin drawn is biased.

∴ P (E_1) = m/N and P (E_2) = N – m/N

A is the event that on tossing the coin head appears first and then appears tail.

∴ P (A) = P ($E_1 \cap A$) + P ($E_2 \cap A$)

= P (E_1) P (A| E_1) + P (E_2) P (A| E_2)

= m/n $(1/2)^2$ + (N – m/N) (2/3) (1/3) .... (1)

We have to find the probability that A has happened because of E_1

$\therefore$ P (E_1| A) = P (E_1| ∩ A)/P (A)

$$= \frac{\frac{m}{n}\left(\frac{1}{2}\right)^2}{\frac{m}{n}\left(\frac{1}{2}\right)^2 + \frac{N-m}{N}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}$$

$$= \frac{m/4}{m/4 + \frac{2(N-m)}{9}} = 9m/m + 8N$$

Sol. 29.

Let us consider

$E_1 \equiv$ event of passing I exam

$E_2 \equiv$ event of passing II exam

$E_3 \equiv$ event of passing III exam

Then a student can qualify in anyone of following ways

1. He passes first and second exam.

2. He passes fires, fails in second but passes third exam.

3. He fails in first, passes second and third exam.

$\therefore$ Required probability

= P (E_1) P (E_2| E_1) + P (E_1) P (E_2| E_1) P (E_3| E_2) + P (E_1) P (E_2| E_1) P (E_3| E_2)

[as an event is dependent on previous one]

= p. p + p. ((1 – p). p/2 + (1 – p) p/2 p

= $p^2 + p^2/2 - p^3/2 + p^2/2 - p^3/2 = 2p^2 - p^3$

Sol. 30.

Let us consider the events

$E_1 \equiv$ A hits B Then P (E_1) = 2/3

$E_2 \equiv$ B hits A P (E_2) = 1/2

$E_3 \equiv$ C hits A P (E_3 = 1/3

$E \equiv A$ is hit

$P(E) = P(E_2 \cup E_3) = 1 - P(\bar{E}_2 \cap \bar{E}_3)$

$= 1\ P(\bar{E}_2)\ P(\bar{E}_3) = 1 - 1/2.\ 2/3 = 2/3$

To find $P(E_2 \cap \bar{E}_2 / E)$

$= P(E_2 \cap \bar{E}_3)/P(E)$ [$\because P(E_2 \cap \bar{E}_3 \cap E) = P(E_2 \cap \bar{E}_3)$ i.e., B hits A is hit = B hits A].

$= P(E_2).\ P(\bar{E}_3)/P(E) = 1/2 \times 2/3 / 2/3 = 1/2$

Sol. 31.

Given that A and B are two independent events. C is the event in which exactly of A or B occurs.

Let $P(A) = x$, $P(B) = y$

Then $P(C) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$P(A)\ P(\bar{B}) + P(\bar{A})\ P(B)$

[$\because$ If A and B are independent so are 'A and $\bar{B}$' and '$\bar{A}$ and B'.]

$\Rightarrow P(C) = x(1 - y) + y(1 - x)$ .... (1)

Now consider, $P(A \cup B)\ P(\bar{A} \cap \bar{B})$

$= [P(A) + P(B) - P(A)\ P(B)]\ [P(\bar{A})\ P(\bar{B})]$

$= (x + y - xy)(1 - x)(1 - y)$

$= (x + y)(1 - x)(1 - y) - xy(1 - x)(1 - y) \leq (x + y)(1 - x)(1 - x)$ [$\because x, y \in (0, 1)$]

$= x(1 - x)(1 - y) + y(1 - x)(1 - y)$

$= x(1 - y) + y(1 - x) - x^2(1 - y) - y^2(1 - x) \leq x(1 - y) + y(1 - x)\ 3 = P(C)$ [Using eq^n (1)]

Thus $P(C) \geq P(A \cup B)\ P(\bar{A} \cap \bar{B})$ is proved.

Sol. 32.

Let us define the following events

$A \equiv 4$ white balls are drawn in first six draws

$B \equiv 5$ white balls are drawn in first six draws

$C \equiv 6$ white balls are drawn in first six draws

$E \equiv$ exactly one white ball is drawn in next two draws (i.e. one white and one red)

Then P (E) = P (E| A) P (A) + P (E| B) P (B) + P (E| C) P (C)

But P (E |C) = 0 [As there are only 6 white balls in the bag.]

P (E) = P (E| A) P (A) + P (E| B) P (B)

$$= \frac{{}^{10}C_1 x {}^{2}C_1}{{}^{12}C_2} \frac{{}^{12}C_2 x {}^{6}C_4}{{}^{18}C_6} + \frac{{}^{11}C_1 x {}^{1}C_1}{{}^{12}C_2} \frac{{}^{12}C_1 x {}^{6}C_5}{{}^{18}C_6}$$

Sol. 33.

Let us define the following events

C ≡ person goes by car,

S ≡ person goes by scooter,

B ≡ person goes by bus,

T ≡ person goes by train,

L ≡ person reaches late

Then we are given in the question

P (C) = 1/7; P (S) = 3/7; P (B) = 2/7 P (T) = 1/7

P (L|C) = 2/9; P (L| S) = 1/9; P (L| B) = 4/9; P (L| T) = 1/9

To find the prob. P (C|$\bar{L}$) [∵ reaches in time ≡ not late] Using Baye's theorem

P (C |$\bar{L}$) = P ($\bar{L}$| C) P (C)/ P ($\bar{L}$| C) P (C) + P ($\bar{L}$| S) P (S) .... (i)

+ P ($\bar{L}$| B) P (B) + A ($\bar{L}$| T) P (T)

Now, P ($\bar{L}$| C) = 1 – 2/9 = 7/9; P ($\bar{L}$| S) = 1 – 1/9 = 8/9

P ($\bar{L}$| B) = 1 – 4/9 = 5/9; P ($\bar{L}$| T) = 1 – 1/9 = 8/9

Substituting these values in eqn. (i) we get

P (C| $\bar{L}$) = 7/9 x 1/7 / 7/9 x 1/7 + 8/9 x 3/7 + 5/9 x 2/7 + 8/9 x 1/7

= 7/7 + 24 + 10 + 8 = 7/49 = 1/7.