

(1) If  $f(x) = x \sin \frac{1}{x}$  for  $x \neq 0$  and  $f(0) = 0$ , prove that  $f$  is continuous but not differentiable at 0.

(2) Find derivatives of the following functions using the definition of a derivative

$$(i) \frac{x-1}{x+1} \quad (ii) a^{3x} \quad (iii) \sin x^2 \quad (iv) x \sin x$$

$$[ \text{Ans: } (i) \frac{2}{(x+1)^2}, (ii) 3a^{3x} \log a, (iii) 2x \cos x^2, (iv) x \cos x + \sin x ]$$

(3) If  $f(x) = x^2 \sin \frac{1}{x}$ ,  $x \neq 0$  and  $f(0) = 0$ , prove that  $f'(0) = 0$ .

(4) If  $f(x) = e^x - 1$ ,  $x \geq 0$  and  $f(x) = \ln x$ ,  $x < 0$ , is  $f$  continuous at 0? Is it differentiable at 0?

[ Ans: continuous, not differentiable ]

Find derivatives with respect to  $x$  of the following functions:

$$(5) \frac{x^2 \sin x}{\log x}$$

$$[ \text{Ans: } \frac{x^2 \log x \cos x + 2x \log x \sin x - x \sin x}{(\log x)^2} ]$$

$$(6) 3^x \log x$$

$$[ \text{Ans: } 3^x (1 + x \log 3) ]$$

$$(7) x \cdot 3^x \sin x$$

$$[ \text{Ans: } 3^x (\log 3 \cdot x^2 \sin x + 2x \sin x + x^2 \cos x) ]$$

$$(8) \log_a (x^n)$$

$$[ \text{Ans: } \frac{1}{x \log a} ]$$

$$(9) \frac{e^x}{\log x}$$

$$[ \text{Ans: } \frac{e^x (x \log x - 1)}{x (\log x)^2} ]$$

$$(10) \log [\log (\log x)]$$

$$[ \text{Ans: } \frac{1}{x \log x \log (\log x)} ]$$

Find derivatives with respect to x of the following functions:

$$(11) \log(x + \sqrt{x^2 + a^2})$$

$$\left[ \text{Ans: } \frac{1}{\sqrt{x^2 + a^2}} \right]$$

$$(12) \sqrt{\frac{1-x}{1+x}}$$

$$\left[ \text{Ans: } \frac{-1}{(1-x)^2 (1+x)^2} \right]$$

$$(13) \sin[\log|\cos(e^x + x^2)|]$$

$$\left[ \text{Ans: } -(e^x + 2x) \tan(e^x + x^2) \cos[\log|\cos(e^x + x^2)|] \right]$$

$$(14) \sin[\cos\{\sin(e^x + 1)\}]$$

$$\left[ \text{Ans: } -e^x \cos[\cos(\sin(e^x + 1))] \cdot \sin[\sin(e^x + 1)] \cdot \cos(e^x + 1) \right]$$

$$(15) e^{\log|\sin x|}$$

$$\left[ \text{Ans: } \cos x \text{ if } \sin x > 0, -\cos x \text{ if } \sin x < 0 \right]$$

$$(16) e^{\tan^2 x} \cdot \sin^2 x$$

$$\left[ \text{Ans: } e^{\tan^2 x} (\sin 2x + 2 \tan^3 x) \right]$$

$$(17) \log|\sin(\tan x^2)|$$

$$\left[ \text{Ans: } 2x \cot(\tan x^2) \sec^2 x^2 \right]$$

$$(18) \sqrt{1 - \sin 2x}, \quad 0 < x < \frac{\pi}{2}$$

$$\left[ \begin{array}{l} \text{Ans: } -\sin x - \cos x \text{ for } 0 < x < \frac{\pi}{4}, \quad \cos x + \sin x \text{ for } \frac{\pi}{4} < x < \frac{\pi}{2}, \\ \text{not differentiable at } x = \frac{\pi}{4} \end{array} \right]$$

$$(19) \sin^{-1} \frac{x}{a}, \quad 0 < |x| < |a|$$

$$\left[ \text{Ans: } \frac{1}{\sqrt{a^2 - x^2}} \text{ for } a > 0, \quad \frac{-1}{\sqrt{a^2 - x^2}} \text{ for } a < 0 \right]$$

Find derivatives with respect to x of the following functions:

(20)  $\sin^{-1} 2x \sqrt{1 - x^2}$ ,  $|x| < 1$

$$\left[ \begin{array}{l} \text{Ans: } \frac{-2}{\sqrt{1-x^2}}, \text{ for } x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right) \\ \frac{2}{\sqrt{1-x^2}}, \text{ for } |x| < \frac{1}{\sqrt{2}}, \text{ not differentiable at } |x| = \frac{1}{\sqrt{2}} \end{array} \right]$$

(21)  $\cos^{-1} (4x^3 - 3x)$

$$\left[ \begin{array}{l} \text{Ans: } \frac{3}{\sqrt{1-x^2}} \text{ for } |x| < \frac{1}{2}, \quad \frac{-3}{\sqrt{1-x^2}} \text{ for } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \\ \text{not differentiable for } |x| = \frac{1}{2} \end{array} \right]$$

(22)  $\sec^{-1} \frac{1}{2x^2 - 1}$ ,  $0 < |x| < \dots$  and  $|x| \neq \frac{1}{\sqrt{2}}$

$$\left[ \begin{array}{l} \text{Ans: } -\frac{2}{\sqrt{1-x^2}} \text{ for } 0 < x < 1 \text{ and } x \neq \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{1-x^2}} \text{ for } -1 < x < 0 \text{ and } x \neq -\frac{1}{\sqrt{2}} \end{array} \right]$$

(23)  $\tan^{-1} \frac{\cos x}{1 + \sin x}$

$$\left[ \text{Ans: } \frac{1}{2} \right]$$

(24)  $\tan^{-1} \left[ \frac{\sqrt{1+x^2} - 1}{x} \right]$

$$\left[ \text{Ans: } \frac{1}{2(1+x^2)} \right]$$

(25)  $\tan^{-1} \left[ \frac{1 - \cos x}{1 + \cos x} \right]^{\frac{1}{2}}$  for  $\pi < x < 2\pi$

$$\left[ \text{Ans: } -\frac{1}{2} \right]$$

(26)  $\cot^{-1} \left[ \frac{\sqrt{1+x^2} - 1}{x} \right]$

$$\left[ \text{Ans: } -\frac{1}{2(1+x^2)} \right]$$

Find derivatives with respect to x of the following functions:

$$(27) \quad \sin^{-1} \frac{2x}{1+x^2}$$

$$\left[ \text{Ans: } \frac{2}{1+x^2} \text{ for } |x| < 1, -\frac{2}{1+x^2} \text{ for } |x| > 1, \text{ not differentiable for } |x| = 1 \right]$$

$$(28) \quad \sec^{-1} \frac{1+x^2}{1-x^2}$$

$$\left[ \text{Ans: } \frac{2}{1+x^2} \text{ for } x \in \mathbb{R}^+ - \{1\}, -\frac{2}{1+x^2} \text{ for } x \in \mathbb{R}^+ - \{-1\}, \text{ not differentiable for } x = 0. \right]$$

$$(29) \quad \cos^{-1}x + \cos^{-1}\sqrt{1-x^2}$$

$$\left[ \text{Ans: } 0 \text{ for } x > 0, \frac{2}{\sqrt{1-x^2}} \text{ for } x < 0, \text{ not differentiable for } x = 0. \right]$$

$$(30) \quad \sin^{-1} \left[ \frac{3 \sin x + 4 \cos x}{5} \right] \quad [\text{Ans: } \pm 1]$$

$$(31) \quad \tan^{-1} \left[ \frac{a \sin x + b \cos x}{a \cos x - b \sin x} \right] \quad [\text{Ans: } 1]$$

$$(32) \quad \tan^{-1} \frac{8x}{1+16x^2}$$

$$\left[ \text{Ans: } \frac{2(1-8x^2)}{(1+16x^2)(1+4x^2)} \right]$$

$$(33) \quad \frac{d}{dx} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}, \quad a > 0$$

$$\left[ \text{Ans: } \sqrt{a^2 - x^2} \right]$$

Solve the following problems as directed:

$$(34) \quad \text{Find } \frac{du}{dv}, \quad \text{if } u = \sin^{-1} \frac{2t}{1+t^2} \text{ and } v = \tan^{-1} \frac{2t}{1-t^2}$$

for (i)  $|t| < 1$  and (ii)  $t > 1$ .

[Ans: (i) 1, (ii) -1]

Solve the following problems as directed:

(35) If  $\frac{d}{dx}(x^n) = nx^{n-1}$  for  $n \in \mathbb{N}$ ,

prove that  $\frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n}x^{\frac{1}{n}-1}$  ( $n \in \mathbb{N}$ ,  $x \in \mathbb{R}^+$ ).

(36) Find  $\frac{dy}{dx}$  if  $\cos(x^2 + y^2) = \log(xy)$ .

$$\left[ \text{Ans: } -\left( \frac{2x^2 \sin(x^2 + y^2) + 1}{2y^2 \sin(x^2 + y^2) + 1} \right) \cdot \frac{y}{x} \right]$$

(37) If  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ , find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .

$$\left[ \text{Ans: } \cot \frac{\theta}{2}, -\frac{1}{4a} \operatorname{cosec}^4 \frac{\theta}{2} \right]$$

(38) If  $x = \cos \theta + \cos 2\theta$  and  $y = \sin \theta + \sin 2\theta$ , find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .

$$\left[ \text{Ans: } \frac{dy}{dx} = -\frac{2 \cos 2\theta + \cos \theta}{2 \sin 2\theta + \sin \theta}, \quad \frac{d^2y}{dx^2} = -\frac{3(3 + 2 \cos \theta)}{(2 \sin 2\theta + \sin \theta)^3} \right]$$

(39) If  $ax^2 + 2hxy + by^2 = 0$ , prove that  $\frac{d^2y}{dx^2} = 0$ .

(40) If  $x = \cos \theta - 2 \cos^3 \theta$ ,  $y = 3 \sin \theta - 2 \sin^3 \theta$ ,  $\theta \neq (2k - 1) \frac{\pi}{4}$ , find  $\frac{d^2y}{dx^2}$ .

$$\left[ \text{Ans: } -\frac{1}{3} \operatorname{cosec}^3 \theta \sec 2\theta \right]$$

(41) Find  $\frac{dy}{dx}$ , if  $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ .

$$\left[ \text{Ans: } -\sqrt{\frac{1-y^2}{1-x^2}} \right]$$

(42) Find  $\frac{d^2y}{dx^2}$ , if  $x = 2 \cos t - \cos 2t$ ,  $y = 2 \sin t - \sin 2t$ ,

$t \neq 2k\pi$  or  $(2k - 1)\frac{\pi}{3}$ ,  $k \in \mathbb{Z}$ .

$$\left[ \text{Ans: } \frac{3}{8} \sec^3 \frac{3t}{2} \operatorname{cosec} \frac{t}{2} \right]$$

Solve the following problems as directed:

(43) Find  $\frac{dy}{dx}$ , if  $x = \frac{3at}{1+t^2}$ ,  $y = \frac{3at^2}{1+t^2}$ .

[Ans :  $\frac{2t}{1+t^2}$ ]

(44) If  $x = a \left( \cos \theta + \log \tan \frac{\theta}{2} \right)$ ,  $y = a \sin \theta$ , find  $\frac{d^2y}{dx^2}$ .

[Ans :  $\frac{1}{a} \sec^4 \theta \sin \theta$ ]

(45) Find  $\frac{d^2y}{dx^2}$ , if  $x = a(1 - \cos \theta)$ ,  $y = a(\theta - \sin \theta)$ .

[Ans :  $\frac{1}{4a} \sec^3 \frac{\theta}{2} \operatorname{cosec} \frac{\theta}{2}$ ]

(46) Find  $\frac{dy}{dx}$ , if  $y = \sin x^x$ .

[Ans :  $x^x (1 + \log x) \cos x^x$ ]

(47) For  $y = (\sin x)^x + x^{\sin x}$ , find  $\frac{dy}{dx}$ .

[Ans :  $(\sin x)^x (\log \sin x + x \cot x) + x^{\sin x} \left( \frac{\sin x}{x} + \cos x \log x \right)$ ]

(48) If  $y = (\sqrt{x})^x + x^{\sqrt{x}}$ , find  $\frac{dy}{dx}$ .

[Ans :  $\frac{1}{2}(\sqrt{x})^x (1 + \log x) + \frac{1}{2}x^{\sqrt{x}} - \frac{1}{2}(\log x + 2)$ ]

(49) Find  $\frac{dy}{dx}$  for  $y = x^{\frac{1}{x}} + (1+x)^{\frac{1}{x}}$ .

[Ans :  $x^{\frac{1}{x}-2} (1 - \log x) + (1+x)^{\frac{1}{x}-1} \cdot \frac{x - (1+x)\log(1+x)}{x^2}$ ]

(50) Find  $\frac{dy}{dx}$ , if  $x^m y^n = (x+y)^{m+n}$ .

[Ans :  $\frac{y}{x}$ ]

Solve the following problems as directed:

(51)  $y = x^{x^x}$ . Find  $\frac{dy}{dx}$ .

[Ans:  $x^{x^x} \cdot x^x - 1 (1 + x \log x + x(\log x)^2)$ ]

(52) If  $\sin y = x \sin(a + y)$ , prove that  $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$ .

(53) If  $y = x^y$ , prove that  $\frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$ ,  $x > 0$ .

(54) Prove that  $f(x) = |x - a|$  is not differentiable only at  $x = a$ . Deduce that  $|x - 2| + |x - 3|$  is not differentiable only at  $x = 2$  and  $x = 3$ .

(55) If  $x = at^2$ ,  $y = 2at$  and  $t \neq 0$ , then prove that  $yy_2 + y_1^2 = 0$ , where

$$y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}.$$

(56) For  $x = \tan t$ ,  $y = \cot t$  prove that  $(1 + x^2)y_2 + 2xy_1 = 2pyy_1$ .

(57) If  $y = e^{m \sin^{-1} x}$ , prove that  $(1 - x^2)y_2 - xy_1 = m^2 y$ , where  $m \neq 0$ .

(58) If  $y = e^x (\cos x + \sin x)$ , prove that  $y_2 - 2y_1 + 2y = 0$ .

(59) If  $y = (x + \sqrt{x^2 + 1})^m$ , prove that  $(1 + x^2)y_2 + xy_1 = m^2 y$ .

(60) If  $y = (\cos^{-1} x)^2$ , prove that  $(1 - x^2)y_2 - xy_1 = 2$ .

(61) If  $y = \sin(m \sin^{-1} x)$ , prove that  $(1 - x^2)y_2 - xy_1 + m^2 y = 0$ .

Solve the following problems as directed:

(62) If  $y = \sin pt$ ,  $x = \sin t$ , prove that  $(1 - x^2)y_2 - xy_1 + p^2 y = 0$ .

(63) If  $y = e^{m \tan^{-1} x}$ , prove that  $(1 + x^2)y_2 + (2x - m)y_1 = 0$ .

(64) If  $2x = y^m + y^{-m}$ , prove that  $(x^2 - 1)y_2 + xy_1 = m^2 y$ .

(65) If  $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ , prove that  $4xy_2 + 2y_1 = 0$ .

(66) If  $\cos^{-1} \frac{y}{b} = \log \left( \frac{x}{n} \right)^n$ , prove that  $x^2 y_2 + xy_1 + n^2 y = 0$ .

Find derivatives with respect to  $x$  of the following functions:

(67)  $x \sin^{-1} \frac{2x}{1+x^2}$ ,  $|x| < 1$

[ Ans :  $\frac{2x}{1+x^2} + 2 \tan^{-1} x$  ]

(68)  $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$

[ Ans :  $\frac{\cos^{-1} x}{(1-x^2)^{\frac{3}{2}}} - \frac{x}{(1-x^2)^2}$  ]

(69)  $\tan \left( \frac{\cos x - \sin x}{\cos x + \sin x} \right)$

[ Ans: -1 ]

(70)  $\frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}}$

[ Ans :  $\frac{(x^3 - 2)\sqrt{x^2 + 1}}{(x^2 + 2x + 3)(2x - 5)^{\frac{3}{2}}} \left[ \frac{3x^2}{x^3 - 2} + \frac{x}{x^2 + 1} - \frac{2(x+1)}{x^2 + 2x + 3} - \frac{3}{2x - 5} \right]$  ]

Find derivatives with respect to x of the following functions:

( 71 )  $(\sin x)^{\log x} + (\log x)^x$

$$\left[ \text{Ans : } (\sin x)^{\log x} \left( \log x \cot x + \frac{\log(\sin x)}{x} \right) + (\log x)^x \left( \frac{1}{\log x} \log \log x \right) \right]$$

( 72 )  $\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[ \sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right]$

$$\left[ \text{Ans : } \frac{1}{a + b \cos x} \right]$$

Solve the following problems as directed:

( 73 ) Let  $f(x)$  be a function satisfying the condition  $f(-x) = f(x)$  for all  $x$ . If  $f'(0)$  exists, find its value.

[ Ans: 0 ]

( 74 ) If  $x = \sec \theta - \cos \theta$  and  $y = \sec^n \theta - \cos^n \theta$ ,

then show that  $(x^2 - 4) \left( \frac{dy}{dx} \right)^2 = (y^2 + 4)n^2$ .

( 75 ) If  $x = \cos \theta$ ,  $y = \sin^3 \theta$ , prove that  $\left( \frac{dy}{dx} \right)^2 + y \left( \frac{d^2y}{dx^2} \right) = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$ .

( 76 ) If  $y^2 = p(x)$ , then prove that  $2 \frac{d}{dx} \left( y^3 \frac{d^2y}{dx^2} \right) = p(x)p'''(x)$ .

( 77 ) If  $u$  and  $v$  are derivable functions of  $x$ ,

then prove that  $\frac{d}{dx}(u^v) = v u^{v-1} \frac{du}{dx} + u^v \frac{dv}{dx} \log u$ .

( 78 ) If  $f(2) = 4$ ,  $g(2) = 9$ ,  $f'(2) = g'(2)$ , then find  $\lim_{x \rightarrow 2} \frac{\sqrt{f(x)} - 2}{\sqrt{g(x)} - 2}$ . [ Ans :  $\frac{3}{2}$  ]

Solve the following problems as directed:

( 79 ) Prove that  $\frac{d^2x}{dy^2} = - \frac{d^2y}{dx^2} \div \left( \frac{dy}{dx} \right)^3$ .

( 80 ) If  $\tan \frac{y}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{x}{2}$ , prove that  $\frac{dy}{dx} = \frac{\sin y}{\sin x} = \frac{\sqrt{1-e^2}}{1+e \cos x}$ .

( 81 ) If  $ky = \sin(x+y)$ , prove that  $y_2 = -y(1+y_1)^3$ .

( 82 ) If  $\log y = \log \sin x - x^2$ , prove that  $y_2 + 4y_1 + (x+3)y = 0$ .

( 83 ) For  $y = \log_7(\log_7 x^4)$ , obtain  $\frac{dy}{dx}$  [ Ans :  $\frac{1}{x(\log 7)(\log x)}$  ]

( 84 ) If  $\frac{x}{x-y} = \log \frac{a}{x-y}$ , prove that  $\frac{dy}{dx} = 2 - \frac{x}{y}$ .

( 85 ) Prove that  $\sqrt{1-x^2} - \sqrt{1-y^2} = a(x-y)$ , ( $a \neq 0$ )  $\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ .

( 86 ) For  $y = \tan^{-1} \frac{3xa^2 - x^3}{a(a^2 - 3x^2)}$ , obtain  $\frac{dy}{dx}$  [ Ans :  $\frac{3a}{a^2 + x^2}$  ]

( 87 ) Prove that  $y = x \log[(ax)^{-1} + a^{-1}] \Rightarrow x(x+1)y_2 + xy_1 = y - 1$ .

( 88 ) If  $y = x \log \left( \frac{x}{a+bx} \right)$ , prove that  $x^3 y_2 = (y - xy_1)^2$ .

( 89 ) If  $y = \sqrt{x+1} + \sqrt{x-1}$ , prove that  $(x^2 - 1)y_2 + xy_1 = \frac{y}{4}$ .

( 90 ) Differentiate  $\sin^{-1} x$  w.r.t.  $x$ ,  $|x| < 1$  using the definition of derivative.

Solve the following problems as directed:

(91) If  $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$ ,  
prove that  $(x^2 - 1)y_2 + xy_1 - n^2y = 0$ .

(92) If  $g(x_1 + x_2) = g(x_1)g(x_2)$  and  $g(x) \neq 0 \quad \forall x \in D_g$  and  $g'(0) = 2$ ,  
then prove that  $g'(x) = 2g(x)$ .

(93) If  $f^{-1} = g$  and  $f'(x) = \frac{1}{1+x^3}$ , then prove that  $g'(y) = 1 + [g(y)]^3$ .

(94) If  $f(a) = 2$ ,  $f'(a) = 1$ ,  $g(a) = -1$ ,  $g'(a) = 2$  then prove that  
 $\lim_{x \rightarrow a} \frac{g(x)f(a) - f(x)g(a)}{x - a} = 5$ .

(95) For  $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$ , prove that  $p^4 + p^3 \frac{d^2 p}{d\theta^2} = a^2 b^2$ .

(96) If  $(a - b \cos y)(a + b \cos x) = a^2 - b^2$ , prove that  $\frac{dy}{dx} = \frac{(a^2 - b^2)^{\frac{1}{2}}}{a + b \cos x}$ .

(97) If  $S_n = a + ax + ax^2 + \dots$  upto  $n$  terms,  
show that  $(1 - x) \frac{d}{dx} S_n = nS_{n-1} - (n - 1)S_n$ .

(98) P.M.  $y = x \sin y \Rightarrow \frac{dy}{dx} = \frac{\sin y}{1 - x \cos y} = \frac{y}{x(1 - x \cos y)} = \frac{\sin^2 y}{\sin y - y \cos y}$ .

(99) If  $y = f(x)$  is one-one and onto, p.t.  $f''(x) = -(f^{-1})''[f'(x)]^3$ .

(100) Find  $\frac{dy}{dx}$  for  $x = e^{\tan^{-1} \left[ \frac{y - x^2}{x^2} \right]}$ .

[Ans :  $x \left( \frac{y^2}{x^4} + 2 \right)$ ]