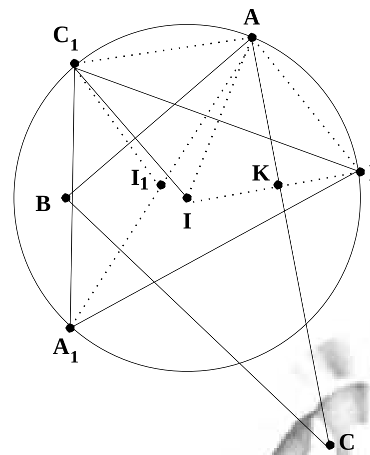
Problems and Solutions of INMO-2008

1. Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

   Solution:

   Note that $IA_1 = IB_1 = IC_1 = 2r$, where r is the in-radius of the triangle ABC. Hence I is the circum-centre of the triangle $A_1B_1C_1$.

   Let K be the point of intersection of IB_1 and AC. Then $IK = r$, $IA = 2r$ and $\angle IKA = 90°$. It follows that $\angle IAK = 30°$ and hence $\angle IAB_1 = 60°$. Thus AIB_1 is an equilateral triangle. Similarly triangle AIC_1 is also equilateral. We hence obtain $AB_1 = AC_1 = AI = IB_1 = IC_1 = 2r$.

   We also observe that $\angle B_1IC_1 = 120°$ and IB_1AC_1 is a rhombus. Thus $\angle B_1AC_1 = 120°$ and by concyclicity $\angle A_1 = 60°$. Since $AB_1 = AC_1$, A is the midpoint of the arc B_1AC_1. It follows that A_1A bisects $\angle A_1$ and I_1 lies on the line A_1A. This implies that

   $$\angle B_1I_1C_1 = 90° + \angle A_1/2 = 90° + 30° = 120°$$

   Since $\angle B_1IC_1 = 120°$, we conclude that B_1, I, I_1, C_1 are concyclic. (Further A is the centre.)

2. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

   Solution: We begin with the standard factorisation

   $$y^4 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2)$$

   Thus we have $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$ for some positive integers m and n such that $m + n = x$. Since $y^2 - 2y + 2 < y^2 + 2y + 2$, we have $m < n$ so that p^m divides p^n. Thus $y^2 - 2y + 2$ divides $y^2 + 2y + 2$. Writing $y^2 + 2y + 2 = y^2 - 2y + 2 + 4y$, we infer that $y^2 - 2y + 2$ divides $4y$ and hence $y^2 - 2y + 2$ divides $4y^2$. But

   $$4y^2 = 4(y^2 - 2y + 2) + 8(y - 1)$$

   Thus $y^2 - 2y + 2$ divides $8(y - 1)$. Since $y^2 - 2y + 2$ divides both $4y$ and $8(y - 1)$, we conclude that it also divides 8. This gives $y^2 - 2y + 2 = 1$, 2, 4 or 8.

   If $y^2 - 2y + 2 = 1$, then $y = 1$ and $y^4 + 4 = 5$, giving $p = 5$ and $x = 1$. If $y^2 - 2y + 2 = 2$, then $y^2 - 2y = 0$ giving $y = 2$. But then $y^4 + 4 = 20$ is not the power of a prime. The equations $y^2 - 2y + 2 = 4$ and $y^2 - 2y + 2 = 8$ have no integer solutions. We conclude that $(p, x, y) = (5, 1, 1)$ is the only solution.

   Alternatively, using $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$, we may get

   $$4y = p^m\left(p^{n-m} - 1\right)$$

   If $m > 0$, then p divides 4 or y. If p divides 4, then $p = 2$. If p divides y, then $y^2 - 2y + 2 = p^m$ shows that p divides 2 and hence $p = 2$. But then $2^x = y^4 + 4$, which shows that y is even. Taking $y = 2z$, we get $2^{x-2} = 4z^4 + 1$. This implies that $z = 0$ and hence $y = 0$, which is a contradiction. Thus $m = 0$ and $y^2 - 2y + 2 = 1$. This gives $y = 1$ and hence $p = 5$, $x = 1$.

3. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

Solution: Suppose $0 \in A$. Then $a^2 = a^2 + 0 \times b$ is rational and $ab = 0^2 + ab$ is also rational for all a, b in A, $a \neq 0$, $b \neq 0$, $a \neq b$. Hence $a = a_1\sqrt{M}$ for some rational a_1 and natural number M. For any $b \neq 0$, we have

$$b\sqrt{M} = \frac{ab}{a_1}$$

which is a rational number.

Hence we may assume 0 is not in A. If there is a number a in A such that $-a$ is also in A, then again we can get the conclusion as follows. Consider two other elements c, d in A. Then $c^2 + da$ is rational and $c^2 - da$ is also rational. It follows that c^2 is rational and da is rational. Similarly, d^2 and ca are also rationals. Thus $d/c = (da)/(ca)$ is rational. Note that we can vary d over A with $d \neq c$ and $d \neq a$. Again c^2 is rational implies that $c = c_1\sqrt{M}$ for some rational c_1 and natural number M. We observe that $c\sqrt{M} = c_1 M$ is rational, and

$$a\sqrt{M} = \frac{ca}{c_1},$$

so that $a\sqrt{M}$ is a rational number. Similarly is the case with $-a\sqrt{M}$. For any other element d,

$$b\sqrt{M} = Mc_1\frac{d}{c}$$

is a rational number.

Thus we may now assume that 0 is not in A and $a + b \neq 0$ for any a, b in A. Let a, b, c, d be four distinct elements of A. We may assume $|a| > |b$. Then $d^2 + ab$ and $d^2 + bc$ are rational numbers and so is their difference $ab - bc$. Writing $a^2 + ab = a^2 + bc + (ab - bc)$, and using the facts $a^2 + bc$, $ab - bc$ are rationals, we conclude that $a^2 + ab$ is also a rational number. Similarly, $b^2 + ab$ is also a rational number.

Consider

$$q = \frac{a}{b} = \frac{a^2 + ab}{b^2 + ab}$$

Note that $a^2 + ab > 0$. Thus q is a rational number and $a = bq$. This gives $a^2 + ab = b^2(q^2 + q)$. Let us take $b^2(q^2 + q) = l$. Then

$$|b| = \sqrt{\frac{l}{q^2 + q}} = \sqrt{\frac{x}{y}},$$

where x and y are natural numbers. Take $M = xy$. Then $|b|\sqrt{M} = x$ is a rational number. Finally, for any c in A, we have

$$c\sqrt{M} = b\sqrt{M}\frac{c}{b},$$

is also a rational number.

4. All the points with integer coordinates in the xy-plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is coloured red and the point $(0, 1)$ is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

Solution: Consider the lattice points(points with integer coordinates) on the lines $y = 0$ and $y = 1$, other than $(0, 0)$ and $(0, 1)$, If one of them, say $A = (p, 1)$, is coloured green, then we have a right-angled triangle with $(0, 0)$, $(0, 1)$ and A as vertices, all having different colours. (See Figures 1 and 2.)

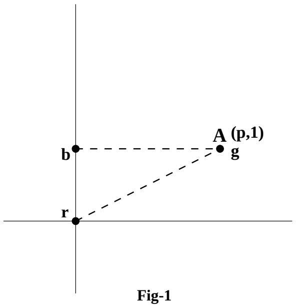


Fig-1

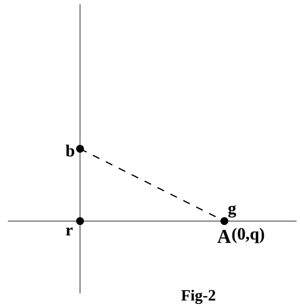


Fig-2

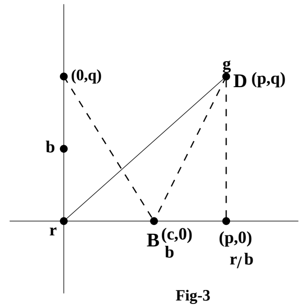


Fig-3

If not, the lattice points on $y = 0$ and $y = 1$ are all red or blue. We consider three different cases.

Case 1. Suppose a point $B = (c, 0)$ is blue. Consider a green point $D = (p, q)$ in the plane. Suppose $p \neq 0$. If its projection $(p, 0)$ on the x-axis is red, then (p, q), $(p, 0)$ and $(c, 0)$ are the vertices of a required type of right-angled triangle. If $(p, 0)$ is blue, then we can consider the triangle whose vertices are $(0, 0)$, $(p, 0)$ and (p, q). If $p = 0$, then the points D, $(0, 0)$ and $(c, 0)$ will work.(Figure 3.)

Case 2. A point $D = (c, 1)$, on the line $y = 1$, is red. A similar argument works in this case.

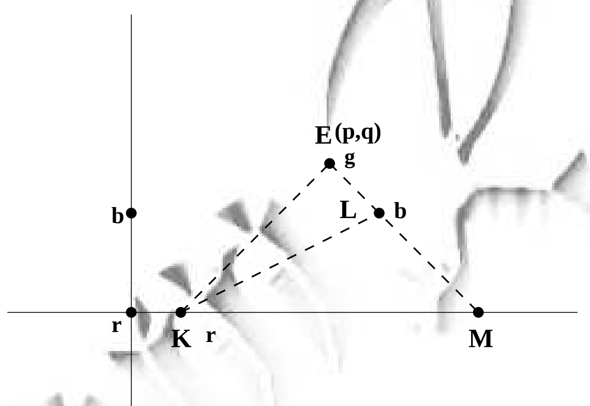
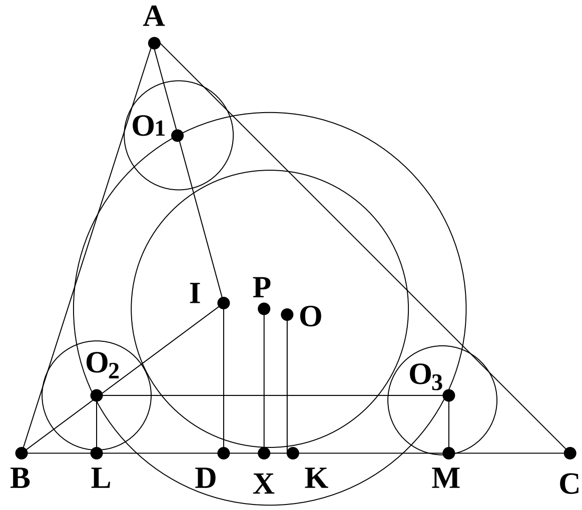


Fig-4

Case 3. Suppose all the lattice points on the line $y = 0$ are red and all on the line $y = 1$ are blue points. Consider a green point $E = (p, q)$, where $q \neq 0$ and $q \neq 1$.(See Figure 4.) Consider an isosceles right-angled triangle EKM with $\angle E = 90°$ such that the hypotenuse KM is a part of the x-axis. Let EM intersect $y =$ in L. Then K is a red point and L is a blue point. Hence EKL is a desired triangle.

5. Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB; and BC, and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

Solution: Let O_1, O_2, O_3 be the centres of the circles Γ_A, Γ_B, Γ_C respectively, and let P be the circum-centre of the triangle $O_1O_2O_3$. Let x denote the common radius of three circles Γ_A, Γ_B, Γ_C. Note that P is also the centre of the circle Γ, as O_1P, O_2P, O_3P each exceed the radius of Γ by x. Let D, X, K, L, M be respectively the projections of I, P, O, O_1, O_2 on BC .



From $\dfrac{BL}{BD} = \dfrac{LO_2}{DI}$, we get $BL = x(s-b)/r$, as $ID = r$ and $BD = (s-b)$. Similarly, $CM = x(s-c)/r$. Therefore, $LM = a - \dfrac{x}{r}(s-b+s-c) = \dfrac{a}{r}(r-x)$. Since O_2LMO_3 is a rectangle and PX is the perpendicular bisector of O_2O_3, it is perpendicular bisector of LM as well. Thus

$$\begin{aligned} LX &= \frac{1}{2}LM = \frac{a}{2r}(r-x); \\ BX &= BL + LX = \frac{x}{r}(s-b) + \frac{a}{2r}(r-x) = \frac{a}{2} - \frac{x(b-c)}{2r}; \\ DK &= BK - BD = \frac{a}{2} - (s-b) = \frac{b-c}{2}; \\ XK &= BK - BX = \frac{a}{2} - \frac{a}{2} + \frac{x(b-c)}{2r} = \frac{x(b-c)}{2r} \end{aligned}$$

Hence we get

$$\frac{XK}{DK} = \frac{x}{r}$$

We observe that the sides of triangle $O_1O_2O_3$ are

$$O_2O_3 = LM = \frac{a}{r}(r-x), \quad O_3O_1 = \frac{b}{r}(r-x), \quad O_1O_2 = \frac{c}{r}(r-x)$$

Thus the sides of $O_1O_2O_3$ and those of ABC are in the ratio $(r-x)/r$. Further, as the sides of $O_1O_2O_3$ are parallel to those of ABC, we see that I is the in-centre of $O_1O_2O_3$ as well. This gives $IP/IO = (r-x)/r$, and hence $PO/IO = x/r$. Thus we obtain

$$\frac{XK}{DK} = \frac{PO}{IO}$$

It follows that I, P, O are collinear.

Alternately, we also infer that I is the centre of homothety which takes the figure $O_1O_2O_3$ to ABC. Hence it takes P to O. It follows that I, P, O are collinear

6. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (i) $P(x)Q(x)$ is a polynomial in x^2; and (ii) $P(x)R(x)$ is a polynomial in x^3.

Solution: Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with integer coefficients.

Part (i) We may write

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big)$$

Define

$$Q(x) = a_0 + a_2x^2 + a_4x^4 + \cdots - x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big)$$

Then $Q(x)$ is also a polynomial with integer coefficients and

$$P(x)Q(x) = \left(a_0 + a_2x^2 + a_4x^4 + \cdots\right)^2 - x^2\left(a_1 + a_3x^2 + a_5x^5 + \cdots\right)^2$$

is a polynomial in x^2.

Part (ii) We write again

$$P(x) = A(x) + xB(x) + x^2C(x),$$

where

$$\begin{aligned} A(x) &= a_0 + a_3x^3 + a_6x^6 + \cdots, \\ B(x) &= a_1 + a_4x^3 + a_7x^6 + \cdots, \\ C(x) &= a_2 + a_5x^3 + a_8x^6 + \cdots \end{aligned}$$

Note that $A(x)$, $B(x)$ and $C(x)$ are polynomials with integer coefficients and each of these is a polynomial in x^3. We may introduce

$$\begin{aligned} S(x) &= A(x) + \omega xB(x) + \omega^2x^2C(x), \\ T(x) &= A(x) + \omega^2xB(x) + \omega x^2C(x), \end{aligned}$$

where ω is an imaginary cube-root of unity. Then

$$S(x)T(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x)$$

since $\omega^3 = 1$ and $\omega + \omega^2 = -1$. Taking $R(x) = S(x)T(x)$, we obtain

$$P(x)R(x) = \left(A(x)\right)^3 + x^3\left(B(x)\right)^3 + x^6\left(C(x)\right)^3 - 3x^3A(x)B(x)C(x),$$

which is a polynomial in x^3. This follows from the identity

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc$$

Alternately, $R(x)$ may be directly defined by

$$R(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x)$$
