Sequences and Series

SUBJECTIVE PROBLEMS:

Q. 1.

The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation. $2A + G^2 = 27$. Find the two numbers. **(IIT JEE -1979 -3Marks)**

Q. 2.

The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°, Find the number of sides of the polygon. **(IIT JEE-1980 -4Marks)**

Q 3.

If $a_1, a_2 \ldots\ldots, a_n$ are in arithmetic progression, where $a_i > 0$ for all I, Show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$ **(IIT JEE-1982-2Marks)**

Q. 4.

Does there exist a geometric progression containing 27, 8 and 12 as three of its terms ? If it exits, how many such progressions are possible ? **(IIT JEE-1982-3Marks)**

Q. 5.

Find three number a, b, c between 2 and 18 such that

(i) their sum is 25

(ii) the numbers 2, a, b sare consecutives terms of an A. P. and

(iii) the numbers b, c, 18 are consecutive terms of a G. P. **(IIT JEE-1983 -2Marks)**

Q 6.

If $a > 0$, $b > 0$ and $c > 0$, prove that

$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$ **(IIT JEE-1984-2Marks)**

Q 7.

If n is a natural such that

$n = p_1^{\alpha 1}. p_2^{\alpha 2}. p_3^{\alpha 3} \ldots\ldots p_k^{\alpha k}$ and $p_1, p_2, \ldots\ldots, p_k$ are distinct primes, then show

that $\ln n \geq k \ln 2$ **(IIT JEE- 1984 -2Marks)**

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Q 8.

Find the sum of the series:

$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[-\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}\ldots\ldots\ldots\ldots \text{ up to m terms}\right]$ **(IIT JEE-1985 -5Marks)**

Q. 9.

The sum of the squares of three distinct real numbers, which are in G. P., is S^2. If their sum is aS, show that

$a^2 \in (\frac{1}{3}, 1) \cup (1, 3)$. **(IIT JEE-1986-5Marks)**

Q. 10.

Solve for x the following equation :

$\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$ **(IIIT JEE -1987-3Marks)**

Q. 11.

If $\log_3 2$, $\log_3 (2_x - 5)$, and $\log_3 (2^x - 7/2)$ are in arithmetic progression, determine the value of x.

(IIT JEE -1990-4Marks)

Q. 12.

Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $(n + 1)/n - 1)^2$ p.

(IIT JEE -1991-4Marks)

Q 13.

If $S_1, S_2, S_3, \ldots\ldots\ldots, S_n$ are the sums of infinite geometric series whose first terms

are 1, 2, 3, ..., n and whose common ratios are 1/2, 1/3, 1/4, ... ,

1/n + 1 respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \ldots\ldots\ldots + S^2_{2n-1}$

(IIT JEE-1991-4Marks)

Q. 14.

The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A. P. Find the intervals in which β and γ lie. **(IIT JEE-1996 -3Marks)**

Q. 15.

Let a, b, c, d be real numbers in G. P. If u, v, w, satisfy the system of equations

$u + 2v + 3w = 6$ **(IIT JEE – 1999 – 10Marks)**

$4u + 5v + 6w = 12$

$6u + 9v = 4$

Then show that the roots of the equations

$(1/u + 1/v + 1/w)x^2 + [(b-c)^2 + (c - a)^2 + (d - b)^2] x + u + v + w = 0$

And $20x^2 + 10(a- d)^2 x - 9 = 0$ are reciprocals of each other.

Q. 16.

The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. **(IIT JEE-2000 - 4Marks)**

Q 17.

Let $a_1, a_2, \ldots\ldots\ldots a_n$ be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be Respectively, the arithmetic mean, geometric mean, and harmonic mean of $a_1, a_2, \ldots\ldots\ldots\ldots a_n$. Find An expression for the geometric mean of $G_1, G_2, \ldots\ldots\ldots\ldots, G_n$ in terms of $A_1, A_2, \ldots\ldots\ldots\ldots, A_n, H_1, H_2, \ldots\ldots\ldots\ldots, H_n$. **(IIT JEE -2001-5 Marks)**

Q. 18.

Let a, b, be positive real numbers. If a, A_1, A_2 b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression,

show that $G_1 G_2/H_1 H_2 = A_1 + A_2/ H_1 H_2 = (2a + b)(a + 2b)/ 9ab$ **(IIT JEE-2002 -5Marks)**

Q. 19.

If a, b, c are in A. P., a^2, b^2, c^2 are in H. P. , then prove that

either $a = b = c$ or a, b, -c/2 form a G. P. **(IIT JEE -2003 -4Marks)**

Q. 20.

If $a_n = 3/4 - (3/4)^2 + (3/2)^2 + \ldots\ldots (-1)^{n-1} (3/4)^n$ and **(IIT JEE- 2000 – 6 Marks)**

$b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \;\forall\; n \geq n_0$.

Sequences and Series-Solutions

SUBJECTIVE PROBLEMS :

Sol. 1.

Let the two numbers be a and b, then

$2ab/a + b = 4$ ....(1); $a + b/2 = A$; $\sqrt{ab} = G$

Also $2A + G^2 = 27 \Rightarrow a + b + ab = 27$ .... (2)

Putting $ab = 27 - (a + b)$ in eqn. (1). We get

$54 - 2(a + b)/a + b = 4 \Rightarrow a + b = 9$ then $ab = 27 - 9 = 18$

Solving the two we gat $a = 6, b = 3$ or $a = 3, b = 6$, which are the required numbers.

Sol. 2.

Let there be in n sides in the polygon.

Then by geometry, sum of all n interior angles of polygon $= (n - 2) * 180°$

Also the angles are in A. P. with the smallest angle = 120° , common difference = 5°

∴ Sum of all interior angles of polygon

$= n/2[2 * 120 + (n - 1) * 5$

Thus we should have

$n/2 [2 * 120 + (n - 1) * 5] = (n - 2) * 180$

$\Rightarrow n/2 [5n + 235] = (n - 2) * 180$

$\Rightarrow 5n^2 + 235n = 360n - 720$

$\Rightarrow 5n^2 - 125n + 720 = 0 \Rightarrow n^2 - 25n + 144 = 0$

$\Rightarrow (n - 16) (n - 9) = 0 \Rightarrow n = 16, 9$

Also if n = 16 then 16^{th} angle $= 120 + 15 * 5 = 195° > 180°$

∴ not possible. Hence n = 9.

Sol. 3.

$a_1, a_2 \ldots\ldots\ldots\ldots a_n$ are in A. P. $\forall a_i > 0$

$\therefore a_1 - a_2 = a_2 - a_3 = \ldots\ldots\ldots\ldots = a_{n-1} - a_n = d$ (a constant) . . . . (1)

Now we have to prove

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots\ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

L. H. S. $= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots\ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \ldots\ldots\ldots\ldots + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$

$= \frac{1}{d}[\sqrt{a_1} - \sqrt{a_2} + \sqrt{a_2} - \sqrt{a_3} + \ldots\ldots\ldots\ldots + \sqrt{a_{n-1}} - \sqrt{a_n}]$

(Using equation (1))

$= \frac{1}{d}[\sqrt{a_1} - \sqrt{a_n}]$

$= \frac{a_1-a_n}{d(\sqrt{a_1}+\sqrt{a_n})} = \frac{(n-1)d}{d\sqrt{a_1}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$

= R H S. Hence Proved.

Sol. 4.

If possible let for a G. P.

$T_p = 27 = AR^{p-1}$ .... (1)

$T_q = 8 = AR^{q-1}$ .... (2)

$T_r = 12 = AR^{r-1}$ .... (3)

From (1) and (2)

$R^{p-q} = 27/8 \Rightarrow R^{p-q} = (3/2)^3$ .... (4)

From (2) and (3) ;

$R^{q-r} = 8/12 \Rightarrow R^{q-r} = (3/2)^{-1}$ .... (5)

From (4) and (5)

$R = 3/2$; $p-q = 3$; $q - r = -1$

$p - 2q + r = 4; p, q, r \in N$ ………… (6)

As there can be infinite natural numbers for p, q, and r to satisfy equation (6)

∴ There can be infinite G. P' s.

Sol. 5.

$2 < a, b, c, < 18 \quad a + b + c = 25$ ……. (1)

2, a, b are AP ⇒ $2a = b + 2$

⇒ $2a - b = 2$ …….. (2)

b, c, 18 are in GP ⇒ $c^2 = 18b$ …… (3)

From (2) ⇒ $a = b + 2/2$

(1) ⇒ $b + 2/2 + b + c = 25 \Rightarrow 3b = 48 - 2c$

(3) ⇒ $c^2 = 6(48 - 2c) \Rightarrow c^2 + 12c - 288 = 0$

⇒ $c = 12, -24$ (rejected)

⇒ $a = 5, b = 8, c = 12$

Sol. 6.

Given that a, b, c > 0

We know for +ve numbers A. M. ≥ G. M.

∴ For +ve numbers a, b, c we get

$a + b + c/3 \geq \sqrt[3]{abc}$ . . . . (1)

Also for +ve numbers 1/a, 1/b, 1/c, we get

$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{abc}}$ . . . . (2)

Multiplying in eqs (1) and (2) we get

$\left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \geq \sqrt[3]{abc} \times \frac{1}{\sqrt[3]{abc}}$

⇒ $(a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ Proved.

Sol. 7.

Given that n = $p_1^{\alpha 1} . p_2^{\alpha 2} .p_3^{\alpha 3}$ . . . . $p^{\alpha k}{}_k$ . . . . (1)

Where n ∈ N and p_1, p_2, p_3 . . . . p_k are distinct

Prime numbers.

Taking log on both sides of eq. (1), we get

log n = α_1 log p_1 + α_2 log p_2 + ... +α_k log p_k . . . . (2)

Since every prime number is such that

$p_i \geq 2$

∴ $\log_e p_i \geq \log_e 2$ . . . . (3)

∀ I = 1 (1) k

Using (2) and (3) we get

log n ≥ α_1 log 2 + α_2 log 2 + α_3 log 2 + ... + α_k log 2

=> log n ≥ ($\alpha_1 + \alpha_2$ + ... + α_k) log 2

=> log n ≥ k log 2 Proved

Sol. 8.

The given series is

$\sum_{r=0}^{n}(-1)^r \, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{2r}} + \frac{15^r}{2^{4r}} + \right.$ . . . . up to m terms]

$\sum_{r=0}^{n}(-1)^r \, {}^nC_r \left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \left(\frac{15}{16}\right)^r +$ . . . . to m terms]

Now, $\sum_{r=0}^{n}(-1)^r \, {}^nC_r \left(\frac{1}{2}\right)^r = 1 - {}^nC_1 . 1/2 + {}^nC_2 . ½^2 - {}^nC_3 . ½^3 +$ . . . .

$= \left(1 - \frac{1}{2}\right)^n = \frac{1}{2^n}$

Similarly, $\sum_{r=0}^{n}(-1)^r \, {}^nC_r \left(\frac{3}{4}\right)^r = \left(1 - \frac{3}{4}\right)^n = \frac{1}{4^n}$ etc.

Hence the given series is, $= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} +$ . . . . to m terms

$= \frac{\frac{1}{2^n}\left(1-\left(\frac{1}{2^n}\right)^m\right)}{1-\frac{1}{2^n}}$ [Summing the G. P.] $= \frac{2^{mn}-1}{2^{mn}(2^n-1)}$

Sol. 9.

Let the three distinct real numbers be $\alpha/r, \alpha, \alpha r$.

Since sum of squares of three numbers be S^2

$\therefore\ \alpha^2/r^2 + \alpha^2 + \alpha^2 r^2 = S^2$

Or $\alpha^2(1 + r^2 + r^4)/r^2 = S^2$ .... (1)

Sum of numbers is aS

$\therefore \alpha (1 + r + r^2)/r = aS$ .... (2)

Dividing eq. (1) by the square (2), we get

$\alpha^2 (1 + r^2 + r^4)/r^2 * r^2/ \alpha^2 (1 + r + r^2)^2 = S^2/a^2S^2$

$(1 + 2r^2 + r^4) - r^2 / (1 + r + r^2)^2 = 1/a^2, (1 + r + r^2) (1 - r + r^2) /(1 + r + r^2)^2 = 1/a^2$

$1 - r + r^2/1 + r + r^2 = 1/a^2 \Rightarrow a^2r^2 - a^2r + a^2 = 1 + r + r^2$

$\Rightarrow (a^2 - 1) r^2 - (a^2 + 1) r + (a^2 - 1) = 0$

$\Rightarrow r^2 + (1 + a^2/1 - a^2)r + 1 = 0$ ...... (3)

For real valuew of r, $D \geq 0$

$\Rightarrow (1 + a^2/1 - a^2)^2 - 4 \geq 0$

$\Rightarrow 1 + 2a^2 + a^2 - 4 + 8a^2 - 4^2 \geq 0$

$\Rightarrow 3a^4 - 10a^2 + 3 \leq 0 \Rightarrow (3a^2 - 1) (a^2 - 3) \leq 0$

$\Rightarrow (a^2 - 1/3) (a^2 - 3) \leq 0$

Clearly the above inequality holds for

$1/3 \leq a^2 \leq 3$

But from eq. (3), $a \neq 1 \therefore a^2 \in (1/3, 1) \cup (1 , 3)$.

Sol. 10.

The given equation is

$\log_{(2x + 3)} (6x^2 + 23x + 21)$

$= 4 - \log_{3x+7} (4x^2 + 12x + 9)$

$\Rightarrow \log_{(2x+3)} (6x^2 + 23x\ 21)$

$+ \log_{(3x+7)} (4x^2 + 12x + 9) = 4$

$\Rightarrow \log_{(2x+3)} (2x + 3)(3x + 7) + \log_{(3x+7)} (2x + 3)^2 x = 4$

$\Rightarrow 1 + \log_{(2x+3)} (3x + 7) + 2 \log_{(3x+7)} (2x + 3) = 4$

[Using log ab = a + log b and log a^n = n log a]

NOTE THIS STEP

$\Rightarrow \log_{(2x+3)} (3x + 7) + 2/ \log_{(2x+3)} (3x + 7) = 3$ [Using $\log_a b = 1/\log_b a$]

Let log $\log_{(2x+3)} (3x + 7) = y$

$\Rightarrow y + 2/y = 3 \Rightarrow y^2 - 3y + 2 = 0$

$\Rightarrow (y - 1)(y - 2) = 0 \Rightarrow y = 1, 2$

Substituting the values of y in (1), we get

$\Rightarrow \log_{(2x+3)} (3x + 7) = 1$ and $\log_{(2x+3)} (3x + 7) = 2$

$\Rightarrow 3x + 7 = 2x + 3$ and $3x + 7 = (2x + 3)^2$

$\Rightarrow x = -4$ and $4x^2 + 9x + 2 = 0$

$\Rightarrow x = -4$ and $(x + 2)(4x + 1) = 0$

$\Rightarrow x = -4$ and $x = 02, x = -1/4$

As $\log_a x$ is defined for $x > 0$ and $a > 0$ ($a \neq 1$), the possible value of x should satisfy all of the following inequalities :

$\Rightarrow 2x + 3 > 0$ and $3x + 7 > 0$

Also $2x + 3 \neq 1$ and $3x + 7 \neq 1$

Out of $x = -4$, $x = -2$ and $x = -1/4$ only $x = -1/4$

Satisfies the above inequalities.

So only solutions is $x = -1/4$.

Sol. 11.

Given that $\log_3 2$, $\log_3(2^x - 5)$, $\log_3 (2^x - 7/2)$ are in A. P.

$\Rightarrow 2 \log_3 (2^x - 5) = \log_3 2 + \log_3 (2^x - 7/2)$

$\Rightarrow (2^x - 5)^2 = 2 (2^x - 7/2)$

$\Rightarrow (2x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 12.2^x + 32 = 0$

Let $2^x = y$, then we get,

$Y^2 - 12y + 32 = 0 \Rightarrow (y - 4)(y - 8) = 0$

$\Rightarrow y = 4$ or $8 \Rightarrow 2^x = 2^2$ or $2^3 \Rightarrow x = 2$ or 3

But for $\log_3(2x - 5)$ and $\log_3 (2x - 7/2)$ to be defined

$2^x - 5 > 0$ and $2^x - 7/2 > 0$

$\Rightarrow 2^x > 5$ and $2^x > 7/2$

$\Rightarrow \quad 2^x > 5$

$\Rightarrow x \neq 2$ and therefore $x = 3$.

Sol. 12.

Let a and b be two numbers and $A_1, A_2, A_3, \ldots\ldots\ldots A_n$ be n A. M's between a and b

Then $a, A_1, A_2, \ldots\ldots. A_n, b$ are in A. P. There are (n + 2) terms in the series, so that

$a + (n + 1)d = b \Rightarrow d = b - a/n + 1$

$\therefore \quad A_1 = a + b - a/n + 1 = an + b/n + 1$

$\therefore \quad$ ATQ $p = an + b/n + 1$ ..... (1)

The first H. M. between a and b, when nHM's are inserted between a and b can be obtained by replacing a by 1/a and b by 1/b in eq. (1) and then taking its reciptocal.

Therefore, $q = 1/(1/a)n + 1/b/n + 1 = (n + 1)ab/bn + a$

$\therefore q = (n + 1)ab/a + bn$ .... (2)

We have to prove that q cannot lie between p and $(n + 1)^2/(n - 1)^2 p$.

Now, $n + 1 > n - 1 \Rightarrow n + 1/n - 1 > 1$

$\Rightarrow (n + 1/n - 1)^2 > 1$ or $p(n + 1/n - 1)^2 > p$

$\Rightarrow\ p < p(n + 1/n - 1)^2$ ………….. (3)

Now to prove the given, we have to show that q is less than p.

For this, let, $p/q = (na + b)(nb + a)/(n + 1)^2 ab$

$\Rightarrow p/q - 1 = n(a^2 + b^2) + ab(n^2 + 1) - (n + 1)^2 ab/(n + 1)^2 ab$

$\Rightarrow p/q - 1 = n(a^2 + b^2 - 2ab)/(n + 1)^2 ab$

$\Rightarrow p/q - 1 = n/(n + 1)^2 (a - b/\sqrt{ab})^2$

$= n/(n + 1)^2 (\sqrt{a/b} - \sqrt{b/a})^2 \Rightarrow p/q - 1 > 0$

$\Rightarrow$ (provided a and b and hence p and q are +ve)

$p > q$ …….. (4)

From (3) and (4), we get,

$q < p < (n + 1/n + 1)^2 p$

$\therefore$ q can not lie between p and $(n + 1/n + 1)^2 p$, if a and b are +ve numbers.

ALTERNATE SOLUTION:

After getting equations (1) and (2) as in the previous method, substitute $b = p(n + 1) - an$ [from (1)] in equation (2) to get $aq + nq[p(n + 1) - an] = (n + 1)a[p(n + 1) - an]$

$\Rightarrow a^2 n(n + 1) + a[q(1 - n^2) - p(n + 1)^2] + npq(n + 1) = 0$

$\Rightarrow na^2 - [(n + 1)p + (n - 1)q]a + npq = 0$

$\Rightarrow D \geq 0$ ($\because$ a is real)

$\Rightarrow [n + 1)p + (n - 1)q]^2 - 4n^2 pq \geq 0$

$\Rightarrow (n - 1)^2 q^2 + \{2(n^2 - 1) - 4n^2\} pq + (n + 1)^2 p^2 \geq 0$

$\Rightarrow\ q^2 - 2\,n^2 + 1/(n - 1)^2 pq + (n + 1/n - 1)^2 p^2 \geq 0$

$\Rightarrow [q - p(n + 1/n - 1)^2 [q - p] \geq 0$

[On factorizing by discriminent method] $\Rightarrow$ q can not lie between p and $p(n + 1/n - 1)^2$.

Sol. 13.

ATQ we have,

$S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots\dots\dots\dots \infty$

$S_2 = 2 + 2.\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + \dots\dots\dots\dots \infty$

. . . .

. . . .

$S_3 = 3 + 3.\frac{1}{4} + 3\left(\frac{1}{4}\right)^2 + \dots\dots\dots\dots \infty$

$S_n = n + n\frac{1}{n+1} + n\left(\frac{1}{n+1}\right)^2 + \dots\dots\dots\dots \infty$

$\Rightarrow S_1 = \frac{1}{1-\frac{1}{2}} = 2$ [using $S_\infty = \frac{a}{1-r}$]

$S_2 = \frac{2}{1-\frac{1}{3}} = 3, \quad S_3 = \frac{3}{1-\frac{1}{4}} = 4,$

$S_n = \frac{n}{1-\frac{1}{n+1}} = (n+1)$

$\therefore S_1^2 + S_2^2 + S_3^2 + \dots\dots\dots S^2_{2n-1}$

$= 2^2 + 3^2 + 4^2 + \dots\dots\dots + (n+1)^2 + \dots\dots\dots + (2n)^2$

NOTE THIS STEP:

$\sum_{r=1}^{2n} r^2 - 1 = \frac{2n(2n+1)(4n+1)}{6} - 1^2$

$= \frac{n(2n+1)(4n+1)3}{3}$

Sol. 14.

Since x_1, x_2, x_3 are in A. P. T. Therefore, let $x_1 = a - d$, $x_2 = a$ and $x_3 = a + d$ and x_1, x_2, x_3 are the roots of

$x^3 - x^2 + \beta x + \gamma = 0$

We have $\sum\alpha = a - d + a + a + d = 1$ .... (1)

$\sum\alpha\beta = (a - d)a + a(a + d) + (a - d)(a + d) = \beta$ ..... (2)

$\alpha\beta\gamma = (a - d)a(a + d) = -\gamma$ .... (3)

From (1), we get, $3a = 1 \Rightarrow a = 1/3$

From (2), we get $3a^2 - a^2 = \beta$

$\Rightarrow 3(1/3)^2 - a^2 = \beta \Rightarrow 1/3 - \beta = d^2$

(NOTE : In this equation we have two variables β and d but we have only one equation. So at first sight it looks that this equation cannot be solved but we know that

$d^2 \geq 0 \ \forall \ d \in R$ then ; β can be solved).

$\Rightarrow 1/3 - \beta \geq 0 \qquad \because d^2 \geq 0$

$\Rightarrow \beta \leq 1/3 \Rightarrow \beta \in (-\infty, 1/3]$

From (3), $a(a^2 - d^2) = -\gamma$

$\Rightarrow 1/3\ (1/9 - d^2) = -\gamma \Rightarrow 1/27 - 1/3d^2 = -\gamma$

$\Rightarrow \gamma + 1/27 = 1/3d^2 \Rightarrow \gamma + 1/27 \geq 0$

$\Rightarrow \gamma \geq -1/27 \Rightarrow -\gamma \in [-1/27, \infty)$

Hence $\beta \in (-\infty, 1/3)$ and $\gamma \in [-1/27, \infty]$

Sol. 15.

Solving the system of equations, $u + 2v + 3w = 6$,

$4u + 5v + 6w = 12$ and $6u + 9v = 4$

We get $u = -1/3$, $v = 2/3$, $w = 5/3$

$\therefore u + v + w = 2$, $1/u + 1/v + 1/w = -9/10$

Let r be the common ratio of the G. P., a, b, c, d. Then $b = ar$, $c = ar^2$, $d = ar^3$.

Then the first equation

$(1/u + 1/v + 1/w)\,x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]\,x + (u + v + w) = 0$

Becomes

$\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]\,x + 2 = 0$

i.e. $9x^2 - 10a^2(1 - r)^2[r^2 + (r + 1)^2 + r^2(r + 1)^2]x - 20 = 0$

i.e. $9x^2 - 10a^2(1 - r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$

i.e. $9x^2 - 10a^2(1 - r)^2(1 + r + r^2)^2 x - 20 = 0$,

i.e. $9x^2 - 10a^2(1 - r^3)^2 x - 20 = 0$ .... (1)

The second equation is

$20x^2 + 10(a - ar^3)^2 x - 9 = 0$

i.e., $20x^2 + 10a^2 (1 - r^3)^2 x - 9 = 0$ .... (2)

Since (2) can be obtained by the substitution $x \rightarrow 1/x$, equations (1) and (2) have reciprocal roots.

Sol. 16.

Let a – 3d, a – d, a + d and a + 3d be any four consecutive terms of an A. P. with common difference 2d $\because$ Terms of A. P. are integers, 2d is also an integer.

Hence $p = (2d)^4 + (a - 3d)(a - d)(a + d)(a + 3d)$

$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2) = a^2 - 5d^2)^2$

Now, $a^2 - 5d^2 = a^2 - 9d^2 + 4d^2$

$= (a - 3d)(a + 3d) + (2d)^2$ = some integer

Thus p = square of an integer.

Sol 17.

Given that $a_1, a_2, \ldots\ldots\ldots a_n$ are +ve real no's is G. P.

$$\left.\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_n &= ar^{n-1} \end{aligned}\right| \text{ As } a_1, a_2, \ldots\ldots a_n \text{ are +ve} \therefore r > 0$$

A_n is A. M. of $a_1, a_2, \ldots\ldots\ldots, a_n$

$\therefore A_n = a_1 + a_2 + \ldots\ldots\ldots + a_n/n = a + ar + \ldots\ldots\ldots + ar^{n-1}/n$

$A_n = \frac{a(1-r)^n}{n(1-r)}$ .... (1) (For $r \neq 1$)

G_n is G. M. of $a_1, a_2, \ldots\ldots\ldots, a_n$

$\therefore G_n = \sqrt[n]{a_1, a_2 \ldots\ldots\ldots a_n} = \sqrt[n]{a.ar.ar^2 \ldots\ldots\ldots ar^n - 1}$

$= \sqrt[n]{a^n, r^{\frac{n(n-1)}{2}}} = ar^{\frac{(n-1)}{2}}$

$G_n = ar^{\frac{(n-1)}{2}}$ .... (2) ($r \neq 1$)

H_n is H. M. of $a_1, a_2 \ldots\ldots\ldots a_n$

$\therefore H_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+ \dots\dots +\frac{1}{a_n}} = \frac{n}{\frac{1}{a}+\frac{1}{ar}+ \dots +\frac{1}{ar^{n-1}}}$

$= \frac{n}{\frac{1}{a}\frac{\left(\frac{1}{r^n}-1\right)}{\left(\frac{1}{r}-1\right)}} = \frac{n}{\frac{1}{a}\left(\frac{1-r^n}{r^n}\right).\frac{r}{1-r}}$

$H_n = an\, r^{n-1} (1 - r)/(1 - r^n)$ $(r \neq 1)$. . . . . (3)

We also observe that

$A_n H_n = \frac{a(1-r^n)}{n(1-r)} x \frac{anr^{n-1}(1-r)}{1-r^n} = a^n r^{n-1} = G_n^2$

$\therefore A_n H_n = G_n^2$ . . . . (4)

$\therefore$ Now, G. M. of $G_1, G_2, \dots\dots G_n$ is

$G = \sqrt[n]{G_1\, G_2 \dots\dots\dots\dots G_n}$

$G = \sqrt[n]{\sqrt{A_1H_1}\,\sqrt{A_2H_2} \dots\dots \sqrt{A_nH_n}}$ [Using (4)

$G = (A_1 A_2 \dots\dots\dots\dots A_n H_1 H_2 \dots\dots\dots\dots H_n)^{1/2n}$ . . . . (5)

If r = 1 then

$A_n = G_n = H_n = a$

Also $A_n H_n = G_2^n$

$\therefore$ For r = 1 also, equation (5) holds.

Hence we get

$G = (A_1 A_2 \dots\dots\dots\dots A_n H_1 H_2 \dots\dots\dots\dots H_n)^{1/2n}$

Sol. 18.

Clearly $A_1 + A_2 = a + b$

$1/H_1 + 1/H_2 = 1/a + 1/b$

$\Rightarrow H_1 + H_2/H_1 H_2 = a + b/ab = A_1 + A_2/G_1 G_2$

$\Rightarrow G_1 G_2 / H_1 H_2 = A_1 + A_2/H_1 + H_2$

Also $1/H_1 = 1/a + 1/3\,(1/b – 1/a) \Rightarrow H_1 = 3ab/2b + a$

$1/H_2 = 1/a + 2/3(1/b – 1/a) \Rightarrow H_2 = 3ab/2a + b$

$\Rightarrow A_1 + A_2/H_1 + H_2 = \frac{a+b}{3ab(\frac{1}{2b+a}+\frac{1}{2a+b})}$

$= \frac{(2b+a)(2a+b)}{9ab}$

Sol. 19.

Given that a, b, c are in A. P.

$\Rightarrow 2b = a + c$ ……. (1)

And a^2, b^2, c^2 are in H. P.

$\Rightarrow \frac{1}{b^2} - \frac{1}{d^2} = \frac{1}{c^2} - \frac{1}{b^2}$

$\Rightarrow (a - b)(a + b)/b^2a^2 = (b - c)(b + c)/b^2c^2$

$\Rightarrow ac^2 + bc^2 = a^2b + a^2c$ [$\because a - b = b - c$]

$\Rightarrow ac(c - a) + b(c - a)(c + a) = 0$

$\Rightarrow (c - a)(ab + bc + ca) = 0$

$\Rightarrow$ either $c - a = 0$ or $ab + bc + ca = 0$

$\Rightarrow$ either $c = a$ or $(a + c)b + ca = 0$ and then form (i) $2b^2 + ca = 0$

Either $a = b = c$ or $b^2 = a(-c/2)$

i.e. a, b, -c/2 are in G. P. Hence Proved

Sol. 20.

$a_n = 3/4 - (3/4)^2 + (3/4)^3 + \ldots\ldots + (-1)^{n-1}(3/4)^n$

$= \frac{\frac{3}{4}(1-(-\frac{3}{4})^n)}{1+\frac{3}{4}} = 3/7\ (1 - (-3/4)^n)$

$B_n = 1 - a_n$ and $b_n > a_n\ \forall\ n \geq n_o$

$\therefore 1 - a_n > a_n \Rightarrow 2a_n < 1$

$\Rightarrow \frac{6}{7}[1 - (-\frac{3}{4})^n] < 1 \Rightarrow -(-\frac{3}{4})^n < 1/6$

$\Rightarrow (-3)^{n+1} < 2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$.

$\therefore$ The least natural no, for which it holds is 6 ($\because$ it holds for every even natural no.)