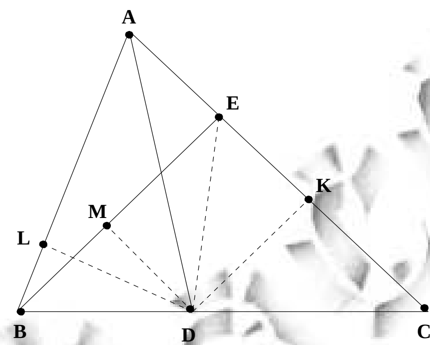
Solutions to CRMO-2007 Problems

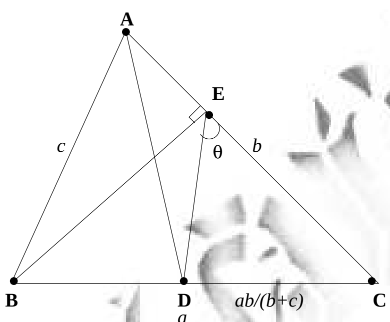
1. Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$.

Solution:

Draw DL perpendicular to AB; DK perpendicular to AC; and DM perpendicular to BE. Then $EM = DK$. Since AD bisects $\angle A$, we observe that $\angle BAD = \angle KAD$. Thus in triangles ALD and AKD, we see that $\angle LAD = \angle KAD$; $\angle AKD = 90° = \angle ALD$; and AD is common. Hence triangles ALD and AKD are congruent, giving $DL = DK$. But $DL > DM$, since BE lies inside the triangle(by acuteness property). Thus $EM > DM$. This implies that $\angle EDM > \angle DEM = 90° - \angle EDM$. We conclude that $\angle EDM > 45°$. Since $\angle CED = \angle EDM$, the result follows.



Alternate Solution:



Let $\angle CED = \theta$. We have $CD = ab/(b+c)$ and $CE = a\cos C$. Using sine rule in triangle CED, we have

$$\frac{CD}{\sin\theta} = \frac{CE}{\sin(C+\theta)}.$$

This reduces to

$$(b+c)\sin\theta\cos C = b\sin C\cos\theta + b\cos C\sin\theta.$$

Simplification gives $c\sin\theta\cos C = b\sin C\cos\theta$ d so that

$$\tan\theta = \frac{b\sin C}{c\cos C} = \frac{\sin B}{\cos C} = \frac{\sin B}{\sin(\pi/2 - C)}.$$

Since ABC is acute-angled, we have $A < \pi/2$. Hence $B + C > \pi/2$ or $B > (\pi/2) - C$. Therefore $\sin B > \sin(\pi/2 - C)$. This implies that $\tan\theta > 1$ and hence $\theta > \pi/4$.

2. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$.

Solution:

We have

$$(c+d)^2 = (a+d)^2 + (b+d)^2.$$

This reduces to

$$d^2 + 2d(a+b-c) + a^2 + b^2 - c^2 = 0.$$

Solving the quadratic equation for d, we obtain

$$d = -(a+b-c) \pm \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)} = -(a+b-c) \pm \sqrt{2(c-a)(c-b)}.$$

Since d is an integer, $2(c-a)(c-b)$ must be a perfect square; say $2(c-a)(c-b) = x^2$, But $\gcd(c-a, c-b) = 1$. Hence we have

$$c-a = 2u^2, \quad c-b = v^2 \quad \text{or} \quad c-a = u^2, \quad c-b = 2v^2,$$

where $u > 0$ and $v > 0$ and $\gcd(u, v) = 1$. In either of the cases $d = -(a+b-c) \pm 2uv$. In the first case

$$c+d = 2c-a-b \pm 2uv = 2u^2 + v^2 \pm 2uv = (u \pm v)^2 + u^2.$$

We observe that $u = v$ implies that $u = v = 1$ and hence $c-a = 2, c-b = 1$. Hence a, b, c are three consecutive integers. We also see that $c+d = 1$ forcing $b+d = 0$, contradicting that $b+d$ is a side of a triangle. Thus $u \neq v$ and hence $c+d$ is the sum of two non-zero integer squares.

Similarly, in the second case we get $c+d = v^2 + (u \pm v)^2$. Thus $c+d$ is the sum of two squares.

Alternate Solution:

One may use charectarisation of primitive Pythagorean triples. Observe that $\gcd(c-a, c-b) = 1$ implies that $c+d, a+d, b+d$ are relatively prime. Hence there exist integers $m > n$ such that

$$a+d = m^2 - n^2, \quad b+d = 2mn, \quad c+d = m^2 + n^2.$$

3. Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

Solution:

Consider the equation $x^2 + ax + b = 0$. It has two roots(not necessarily real), say α and β. Either $\alpha = \beta$ or $\alpha \neq \beta$.

Case 1:

Suppose $\alpha = \beta$, so that α is a double root. Since $\alpha^2 - 2$ is also a root, the only possibility is $\alpha = \alpha^2 - 2$. This reduces to $(\alpha+1)(\alpha-2) = 0$. Hence $\alpha = -1$ or $\alpha = 2$. Observe that $a = -2\alpha$ and $b = \alpha^2$. Thus $(a, b) = (2, 1)$ or $(-4, 4)$.

Case 2:

Suppose $\alpha \neq \beta$. There are four possibilities; (I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$; (II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$; (III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$; or (IV)$\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$

(I) Here $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$. Hence $(a, b) = \big(-(\alpha+\beta), \alpha\beta\big) = (-1, -2)$.

(II) Suppose $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$. Then

$$\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha).$$

Since $\alpha \neq \beta$, we get $\beta + \alpha = -1$. However, we also have

$$\alpha + \beta = \beta^2 + \alpha^2 - 4 = (\alpha+\beta)^2 - 2\alpha\beta - 4.$$

Thus $-1 = 1 - 2\alpha\beta - 4$, which implies that $\alpha\beta = -1$. Therefore $(a,b) = \big(-(\alpha+\beta), \alpha\beta\big) = (1,-1)$.
(III) If $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$, then $\alpha = -\beta$. Thus $\alpha = 2$, $\beta = -2$ or $\alpha = -1$, $\beta = 1$. In this case $(a,b) = (0,-4)$ and $(0,-1)$.
(IV) Note that $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is identical to (III), so that we get exactly same pairs (a,b).
Thus we get 6 pairs; $(a,b) = (-4,4)$, $(2,1)$, $(-1,-2)$, $(1,-1)$, $(0,-4)$, $(0,-1)$.

4. How many 6-digit numbers are there such that:

   (a) the digits of each number are all from the set $\{1,2,3,4,5\}$;

   (b) any digit that appears in the number appears at least twice?

   (Example: 225252 is an admissible number, while 222133 is not.)

Solution:
Since each digit occurs at least twice, we have following possibilities:
1. Three digits occur twice each. We may choose three digits from $\{1,2,3,4,5\}$ in $\binom{5}{3} = 10$ ways. If each occurs exactly twice, the number of such admissible 6-digit numbers is

$$\frac{6!}{2!\,2!\,2!} \times 10 = 900.$$

2. Two digits occur three times each. We can choose 2 digits in $\binom{5}{2} = 10$ ways. Hence the number of admissible 6-digit numbers is

$$\frac{6!}{3!\,3!} \times 10 = 200.$$

3. One digit occurs four times and the other twice. We are choosing two digits again, which can be done in 10 ways.The two digits are interchangeable. Hence the desired number of admissible 6-digit numbers is

$$2 \times \frac{6!}{4!\,2!} \times 10 = 300.$$

4. Finally all digits are the same. There are 5 such numbers.
Thus the total number of admissible numbers is $900 + 200 + 300 + 5 = 1405$.

5. A trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M, and $OM = 2$.

   (a) If $\angle AMB$ is $60°$, determine, with proof, the difference between the lengths of the parallel sides.

   (b) If $\angle AMD$ is $60°$, find the difference between the lengths of the parallel sides.

Solution:
Suppose $\angle AMB = 60°$. Then AMB and CMD are equilateral triangles. Draw OK perpendicular to BD.(see Fig.1) Note that OM bisects $\angle AMB$ so that $\angle OMK =$

$30°$. Hence $OK = OM/2 = 1$. It follows that $KM = \sqrt{OM^2 - OK^2} = \sqrt{3}$. We also observe that

$$AB - CD = BM - MD = BK + KM - (DK - KM) = 2KM,$$

since K is the mid-point of BD. Hence $AB - CD = 2\sqrt{3}$.

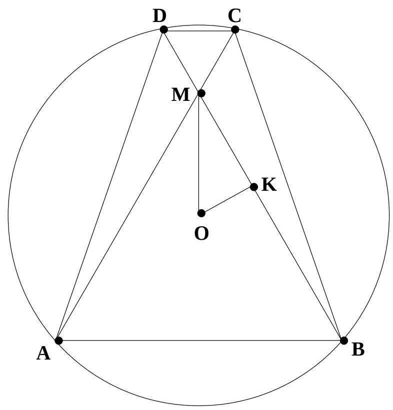


Fig. 1

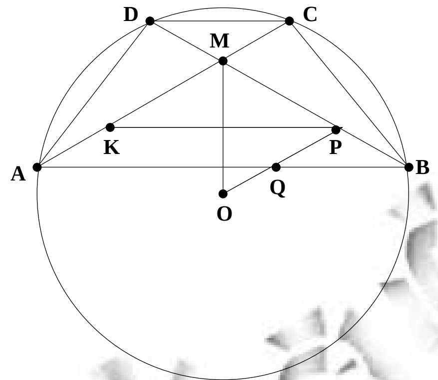


Fig. 2

Suppose $\angle AMD = 60°$ so that $\angle AMB = 120°$. Draw PQ through O parallel to AC (with Q on AB and P on BD). (see Fig.2) Again OM bisects $\angle AMB$ so that $\angle OPM = \angle OMP = 60°$. Thus OMP is an equilateral triangle. Hence diameter perpendicular to BD also bisects MP. This gives $DM = PB$. In the triangles DMC and BPQ, we have $BP = DM$, $\angle DMC = 120° = \angle BPQ$, and $\angle DCM = \angle PBQ$(property of cyclic quadrilateral). Hence DMC and BPQ are congruent so that $DC = BQ$. Thus $AB - DC = AQ$. Note that $AQ = KP$ since $KAQP$ is a parallelogram. But KP is twice the altitude of triangle OPM. Since $OM = 2$, the altitude of OPM is $2 \times \sqrt{3}/2 = \sqrt{3}$. This gives $AQ = 2\sqrt{3}$.

Alternate Solution:

Using some trigonometry, we can get solutions for both the parts simultaneously. Let K, L be the mid-points of AB and CD respectively. Then L, M, O, K are collinear (see Fig.3 and Fig.4). Let $\angle AMK = \theta(= \angle DML)$, and $OM = d$. Since AMB and CMD are similar triangles, if $MD = MC = x$ then $MA = MB = kx$ for some positive constant k.

Now $MK = kx\cos\theta$, $ML = x\cos\theta$, so that $OK = |kx\cos\theta - d|$ and $OL = x\cos\theta + d$. Also $AK = kx\sin\theta$ and $DL = x\sin\theta$. Using

$$AK^2 + OK^2 = AO^2 = DO^2 = DL^2 + OL^2,$$

we get

$$k^2x^2\sin^2\theta + (kx\cos\theta - d)^2 = x^2\sin^2\theta + (x\cos\theta + d)^2.$$

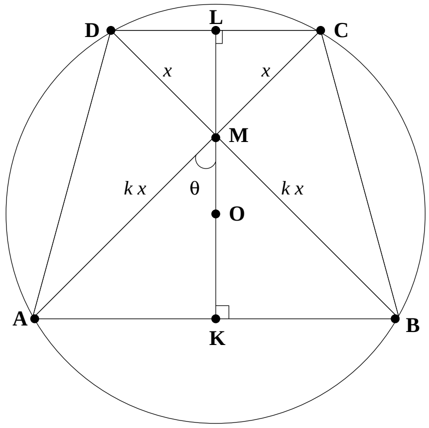


Fig. 3

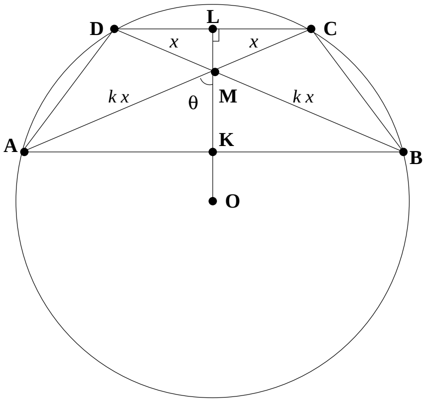


Fig. 4

Simplification gives

$$(k^2 - 1)x^2 = 2xd(k+1)\cos\theta.$$

Since $k + 1 > 0$, we get $(k-1)x = 2d\cos\theta$. Thus

$$\begin{aligned} AB - CD = 2(AK - LD) &= 2(kx\sin\theta - x\sin\theta) \\ &= 2(k-1)x\sin\theta \\ &= 4d\cos\theta\sin\theta \\ &= 2d\sin 2\theta. \end{aligned}$$

If $\angle AMB = 60°$, then $2\theta = 60°$. If $\angle AMD = 60°$, then $2\theta = 120°$. In either case $\sin 2\theta = \sqrt{3}/2$. If $d = 2$, then $AB - CD = 2\sqrt{3}$, in both the cases.

6. Prove that:

   (a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;

   (b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;

   (c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solution:

We have $(2.2)^2 = 4.84 < 5$, so that $\sqrt{5} > 2.2$. Hence $\sqrt[4]{5} > \sqrt{2.2} > 1.4$, as $(1.4)^2 = 1.96 < 2.2$. Therefore $\sqrt[3]{5} > \sqrt[4]{5} > 1.4$. Adding, we get

$$\sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} > 2.2 + 1.4 + 1.4 = 5.$$

We observe that $\sqrt{3} < 3$, $\sqrt[3]{8} = 2$ and $\sqrt[4]{8} < \sqrt[3]{8} = 2$. Thus

$$\sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8} < 3 + 2 + 2 = 7 < 8.$$

Suppose $n \geq 9$. Then $n^2 \geq 9n$, so that $n \geq 3\sqrt{n}$. This gives $\sqrt{n} \leq n/3$. Therefore $\sqrt[4]{n} < \sqrt[3]{n} < \sqrt{n} \leq n/3$. We thus obtain

$$\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} < (n/3) + (n/3) + (n/3) = n.$$